Strategic Payments in Financial Networks

Motivation: Financial Crisis in 2008:

Cascading defaults and bancruptcies

Research on Financial Networks:

- Analyse debts as a major source of risk in financial systems.
- Understand the effects and design suitable measures for regulation of financial systems to avoid cascading insolvency.

The model: (Money Flow Game)

Eisenberg & Noe model of financial networks:

- Set V of financial institutions / firms
- Set E of directed edges, edge e has integral value c_e $\to$ edge $e=(u,v)$ represents a debt of c_e that u owes to v.
- Each firm has liquid assets of $a_v^l \in \mathbb{N}$

The stategies of the nodes:

- Each firm chooses a flow function $f_e: \mathbb{N} \to \mathbb{N}$ for every outgoing edge. This specifies ~~…~~ how assets are distributed.
- f_e fulfills
  - $0 \leq f_e(y) \leq c_e \;\; \forall e \in E^+(v), y \in \mathbb{N}$ (cap. constr.)
  - $f_e(y) \leq f_e(z) \;\; \forall e \in E^+(v)$ and $0 \leq y \leq z$. (non-decreasing)
  - $\Sigma_{e \in E^+(v)} f_e(y) = \min\{y, \ell(v)\}$ (no-fraud constr.)

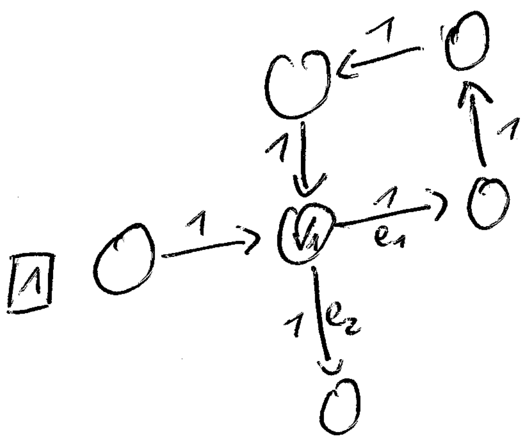
Note: $\ell(v) := \sum_{e \in E^+(v)} c_e$ (liabilities of v).

Given the strategy choices f_e of the players, a clearing state $a = (a_v)_{v \in V}$ is a vector of assets that obeys the strategy choices of the firms, i.e.,

$$a_v = a_v^\ell + \sum_{e=(u,v) \in E^+(v)} f_e(a_u).$$

<u>Idea:</u> The firms strategically seek to maximize their assets, i.e., they try to clear as much ~~t~~ debt as possible.

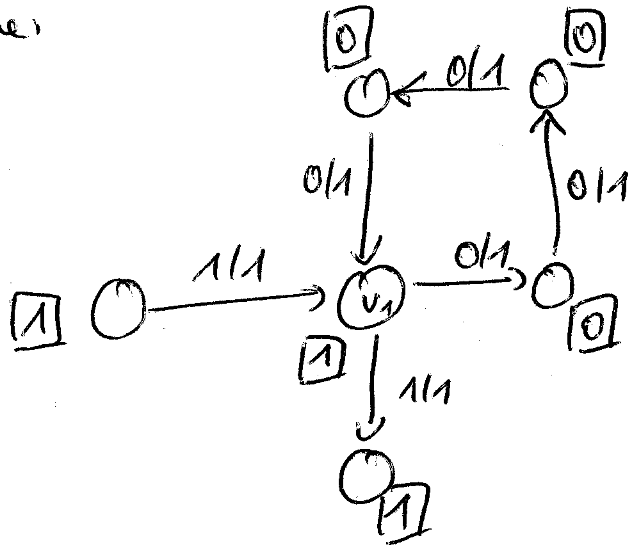
Example:



<u>Case 1.</u> v_1 choose the following strategy:

$f_{e_1}(1) = 0, \quad f_{e_2}(1) = 1.$

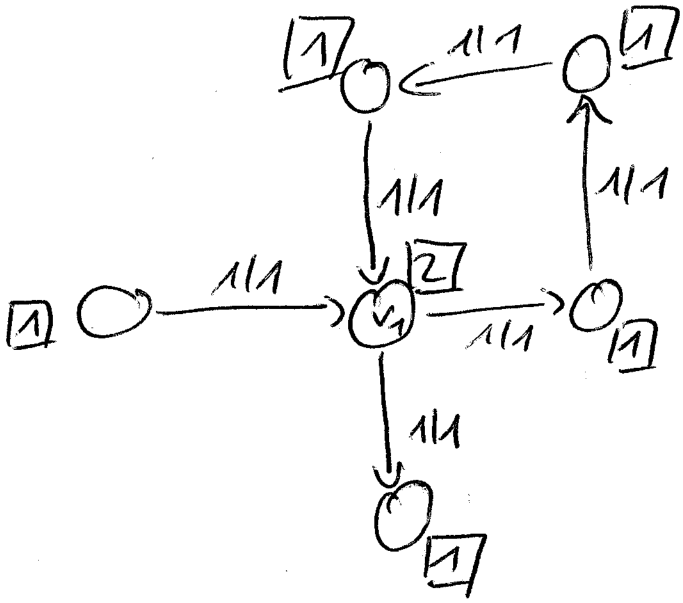
Outcome:



Case 2: v_1 chooses:

$$f_{e_1}(1) = 1 \quad , \quad f_{e_2}(1) = 0 .$$

Outcome:



We consider money flow games in two variants:

Edge-Ranking Games: Each player $v \in V$ pays debts according to a strict and total order over $E^+(v)$. We represent this by a permutation $\pi_v = (e_1, e_2, \ldots)$ over the edges in $E^+(v)$. Here, v pays all debt of e_1, then e_2, etc until all is payed or v runs out of asset.

Coin-Ranking Games Integrality of c_e and a_v^l:

Can replace each edge e in E by c_e many parallel edges of weight 1. $\Rightarrow$ coin-ranking can be represented by edge-ranking in the expanded game.

$\Rightarrow$ We can also use notation $\pi_v = \ldots$, but this might be a pseudo-polynomial blow-up.

Calculating Clearing States.

From a graph $G=(V,E)$ and coin-rankings π,

1. Construct a network G' ~~with flow functions f_e' as as follows.~~ and coin-rankings π' as follows:

   - add an auxiliary node s
   - $\forall v \in V$: add edge (v,s) with $c_{(v,s)} = \infty$
   - $\forall v \in V$ with $a_v^l > 0$: add edge (s,v) with $c_{(s,v)} = a_v^l$, set $a_v^l = 0$.
   - Fix π_s arbitrarily
   - Extend π_v to π_v' by letting v choose (v,s) with least priority

2. Consider a firm $v \in V'$ and some amount of k coins that have already been paid by v. We call the edge for the next coin the *active* edge. (nodes $v \in V$ always have one active edge. s has one or none.)
   Start with $a \equiv 0$. Active edges form disjoint cycles with attached trees. Call

$$o(C) = \{ v \in V \mid \exists \text{ } v\text{-}u\text{-path of active edges for some } u \in C \}$$

   be the orbit of C. Note: if $a_s < l(s)$, every node is in some orbit.

3. Note: $a \equiv 0$ is feasible in G' but not in G, since $a_s = l(s)$ is needed for a 1:1 corresp. to a clearing state in G.
   WHILE $a_s < l(s)$, ~~there is an active edge at s~~, s is in some orbit $o(C)$. Flow conservation => some flow must reach C.
   => flow of at least $\delta_C = \min \{ c(e) \mid e \in C \}$ must be present on every edge of C. => assign this flow & continue.
   Once $a_s = l(s)$, s has no outgoing edge and we have constr. a feasible clearing state in G.

<u>Note:</u> Orbits present at the same time are disjoint.

$\Rightarrow$ Pushing flow along C_1 does not change anything in C_2.

$\Rightarrow$ ~~old~~ averaging state is ind. on the chosen strat. π_s.

$\Rightarrow$ Existence of Clearing State.

<u>Optional Cycles</u> Suppose there is some cycle C, not attached to s.

Pushing flow along these cycles does not hurt feasibility of clearing states in G,

$\Rightarrow$ There can be multiple clearing states.

<u>Can show:</u>

<u>Thm</u> The set of clearing states form a lattice.

(every set of nodes have unique max & unique min)

For the rest of the lecture, we always choose the unique maximum clearing state $\hat{a}$.

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Thm Every money flow game has a strong Equilibrium, SPoS = 1.

Proof: Consider Circulation network above. An optimal circulation f^* (maxim. total flow value) saturates all outgoing edges from s.

$$\Rightarrow \sum_{e \in E'} f_e^* = 2 \cdot \sum_{v \in V} a_v^l + \sum_{e \in E} f_e^* = \sum_{v \in V} a_v^l + \sum_{v \in V} a_v^*$$

$\Rightarrow$ It maximizes assets in G.

Tardos 1985: This can be computed in poly-time. Since all edges are integral: f_e^* integral.

Turn this circulation into a clearing state of some strat. profile.

Threashold ranking strategy: $\pi_v^t = (\pi_v, \tau_v)$. π_v: permut. over $E^+(v)$, τ_v: vector of threasholds $\tau_v = (\tau_e)_{e \in E^+(v)}$ with $0 \le \tau_e \le c_e$.

Interpretation: v pays τ_e to every edge e, in the order given by π_v. Then, v pays remaining coins to every edge until it is full, in the order given by π_v.

Every v chooses an arbitrary order π_v. Choose threasholds $\tau_e = f_e^*$.

Prove that π is strong eq: Let C be a coalition that has a profitable deviation, such that $a_v' > \hat{a}_v \; \forall v \in C$. Consider some $v \in C$.

(a_v': new assets.)

v must have some edge $e=(u,v)$ with more incoming flow. Consider u.

Case 1: $u \in C$. Continue as above.

Case 2: $u \notin C$. $\Rightarrow$ u plays threshold ranking strat. This is monotone

$\Rightarrow$ higher outflow on some edge means higher inflow on some edge.

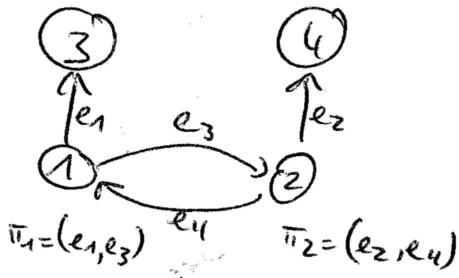
Repeat this argument until we found a cycle of edges with more flow. ↯ f^* optimal circulation.

Thm

$$PoA = \infty$$

Proof:

unit weight edges



Nash with revenue 0. Opt has 2.

Some results on Edge-Ranking games:

- There are games without a ~~f~~ pure Nash equilibrium.
- There is a game with strong PoS of at least $\frac{n}{2} - \varepsilon$. $(n = |V|)$.
- It is NP-hard to decide if there is a strat. profile with total revenue $\geq k$.

$\Rightarrow$ restriction to edge-ranking games is harmful

This motivates the implementation of a solution by some central authority (government, central bank etc.)!