Competitive Routing over Time

(Hoefer, Mirrokni, Röglin, Teng, 2011)

Reminder: Unweighted Network Congestion Game with linear latencies

- directed graph $G=(V,E)$
- set of players $\mathcal{N}=\{1,\dots,n\}$, each player i with
  - a source node $s_i \in V$,
  - a target node $t_i \in V$,
- cost functions $d_e(x) = a_e x$ for $\frac{1}{a_e} \hat{=}$ speed of edge e.
- strategy spaces: Σ_i: set of all simple paths from s_i to t_i in G.

(• if $s_1=s_2=\dots=s_n$, the game is called single-source)

Temporal Network Congestion Game:

As above, with the differece that players do not allocate all edges of the chosen strategy at the same time, but consecutively.

Formally: At each point in time $\tau \in \mathbb{R}_{\geq 0}$, every task (=player) i is located at one edge $e_i(\tau)$ of its chosen path. Fraction $f_i(\tau)$ still unprocessed.

Coordination Mechanism: Decides for each $e \in E$, which task to process.

If e works on i from τ to $\tau+\Delta\tau$ then

$$f_i(\tau+\Delta\tau) = \max\{0, f_i(\tau) - \frac{\Delta\tau}{a_e}\}$$

$\Rightarrow$ task i needs a_e to finish edge e.

Once $f_e^i(t) = 0$, task i arrives at next edge and becomes available.

Coordination Mechanism can only use local information such as weights, arrival times of tasks that have already arrived.

Individual cost of player i in state s: arrival time of i at t.

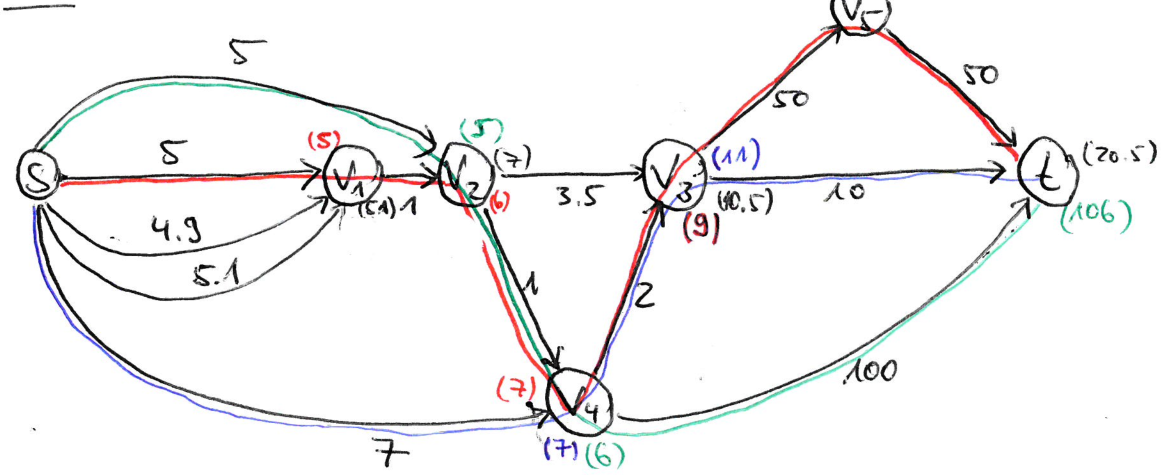
1st Coordination Mechanism: Fifo.

Always execute the task that arrived first non-preemptively. In case of ties, use some global order.

Example:

4 players, global order: 1, 2, 3, 4.

Case 1:

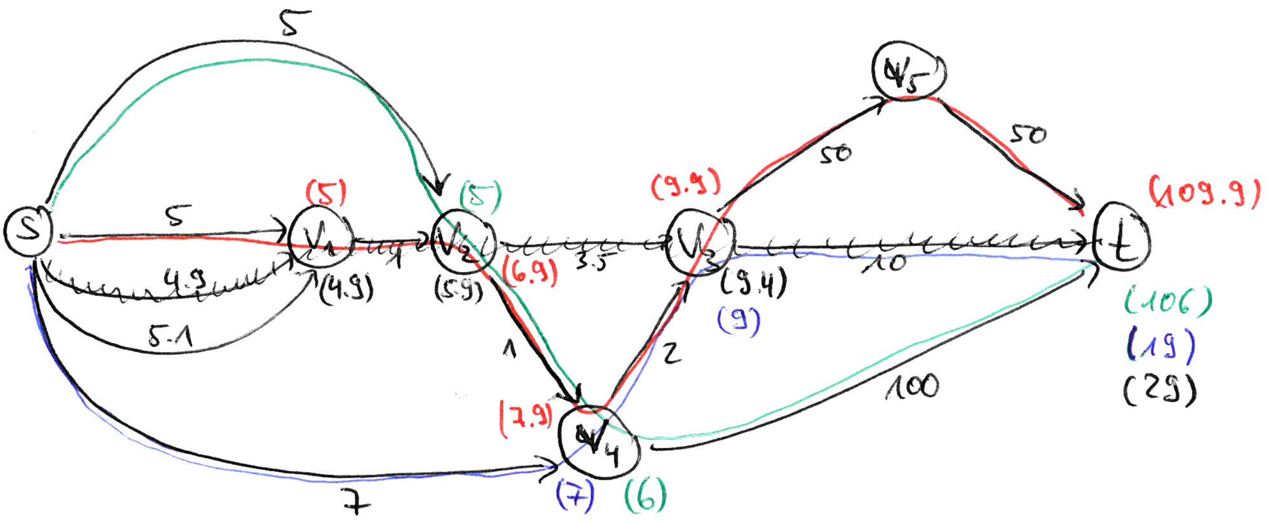


What is the best resp. of P4?

Case 1: s, v1, v2, v3, t with (s, v1) of $a_e = 5.1$: Arrival at time 20.5.

Case 2: s, v1, v2, v3, t with (s, v1) with $a_e = 4.9$:

Case 2:



⇒ Subpaths of a shortest path are not necessarily shortest paths!

If subpaths are shortest paths → greedy best response.

Theorem: For unweighted temporal network congestion games with FIFO, a Nash eq. always exists. It can be computed efficiently.

Proof: W.l.o.g. assume tie-breaking is 1,2,3,…,n. Start in arbitrary state $S=(S_1,S_2,\dots,S_n)$ and let players $1,\dots,n$ play greed best responses in this order. Let $\tilde{S}_1,\dots,\tilde{S}_n$ be the chosen paths.

Claim: In each intermediate state $(\tilde{S}_1,\tilde{S}_2,\dots,\tilde{S}_i,S_{i+1},\dots,S_n)$ the current path $\tilde{S}_j$ is a best response for all $j\leq i$ and nobody of these players can be delayed by a lower ranked player $k>i$.

Proof of Claim: Induction. $i=0$. ⇒ trivially true. $i>0$:

Construct distance function $d: V\to\mathbb{R}_{\geq 0}$ which tells as the earliest possible arrival time of a greedy best resp. of player i. Let I be the set of nodes that have an already assigned distance. Start: $I=\{s\}$, $d(s)=0$. To extend I: Players $1,\dots,i-1$ can not be delayed by others ⇒ every $e\in E$ has a fixed schedule for players $1,\dots,i-1$.

In a greedy best resp, i can not be delayed by players $i+1,\dots,n$. (If i is delayed, there is a faster path to get there.)

$\Rightarrow$ Extend I analogously to Dijkstra's algo:

Insert a node $v \in V \setminus I$ that minimizes $\min_{u\in I} d(u)+\ell(u,v)$, where $\ell(u,v)$ is the time for i to get from u to v, if he arrives at u at time $d(u)$. New distance $d(v) = \min_{u \in I} d(u) + \ell(u,v)$.

(fishes proof of claim)

Claim $\Rightarrow$ Nash eq and polynomial running time.

Theorem: There exist (non-single-source) unweighted temporal congestion games, without pure Nash equilibrium.

Theorem: Computing best responses in unweighted single-source temporal congestion games is NP-hard.

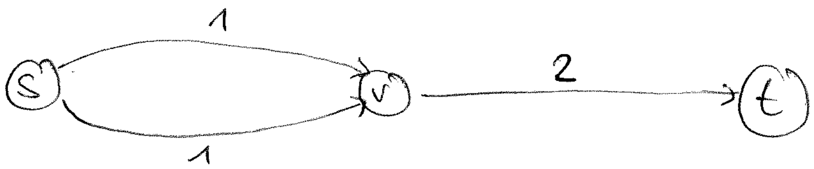
(both without proof)

Extention to weighted temporal congestion games:

- Each player i has a weight $w_i \geq 1$.
- total time on edge e: $a_e w_i$.

Theorem: There is a weighted temporal congestion game with FIFO and without pure Nash equilibrium.

Proof:



2 players: $w_1 = 2$, $w_2 = 3$. Let player 2 have higher priority.

Matching Pennies: If both use the same edge from s to v, player 1 has an incentive to switch, otherwise player 2 has an incentive to switch. □

Another Coordination Mechanism: Non-preemptive global ranking.

- global ranking $\pi : \{1, \dots, n\} \to \{1, \dots, n\}$ with $\pi(1)$ being the task with highest priority.
- Tasks are scheduled non-preemptively according to this ranking. (When an edge e becomes available, the highest ranked task among those that are located at e is processed non-preemptively. It exclusively uses e for a_e time units.)
- Wlog assume $w_1 \leq w_2 \leq \dots \leq w_n$.

a) Shortest - First - Policy.

$\pi(i) = i$.

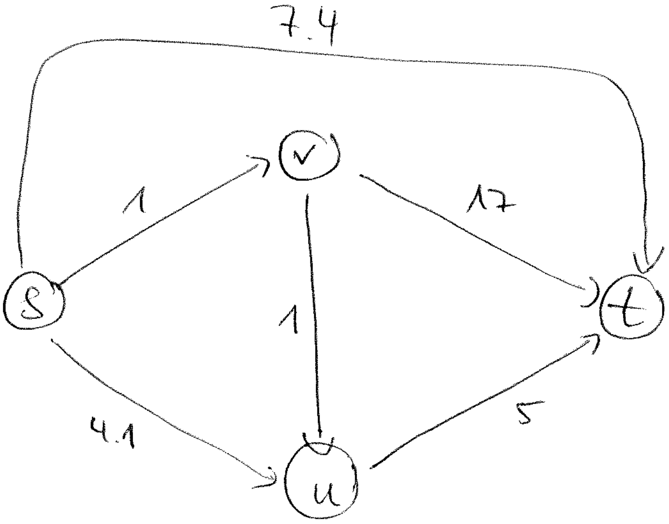
Theorem: In every single-source temporal network congestion game with the shortest-first policy, a pure Nash eq. exists. It can be computed efficiently

Proof idea: Analogously to the proof with FIFO. Again, once all players $1, \dots, i$ play a greedy best-response, they cannot be affected by lower ranked players.

Theorem: Computing a best-resp. is NP-hard.

Theorem: The temporal network cong. game with shortest-first policy is not a potential game.

Proof:



2 players: $w_1 = 1$, $w_2 = 2$:

cycle of improvement steps

$$\begin{tikzcd}
((s,v,t),(s,v,u,t)) \arrow[r] & ((s,v,t),(s,t)) \arrow[d] \\
((s,u,t),(s,v,u,t)) \arrow[u] & ((s,u,t),(s,t)) \arrow[l]
\end{tikzcd}$$

b) Other global ranking:

Theorem: For any given set of players with weights $w_1 \le \dots \le w_n$ and a ranking π different from the identity, there is a temporal congestion game without Nash equilibrium.

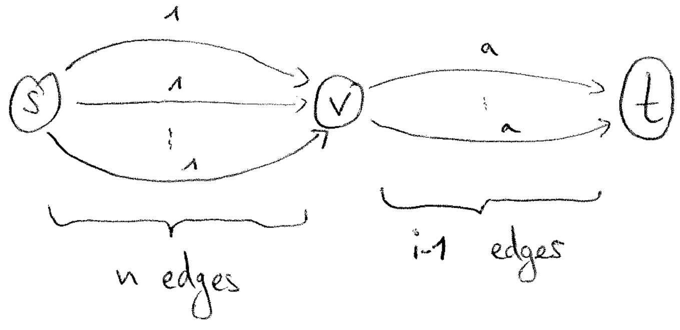
Proof: Let j be such that

$$\pi(1)=1,\ \pi(2)=2, \dots, \pi(j-1)=j-1.$$

and $\pi(j) \neq j$. Then, let $i > j$ be such that $\pi(i)=j$.

Note: $w_i > w_j$.

Consider the following graph:



(a suff. large)

In any Nash eq, players $1,\dots,j-1$ do not have to wait. Otherwise they could switch to a non-used s-v edge. (They have smallest weights)

If one of the players $j,\dots,i-1$ share an edge on the first level with a player with higher prio, they will change to an unused edge. ⇒ Players $1,\dots,i-1$ are the first to arrive at ⓥ in a Nash eq. ⇒ i has to wait. ⇒ i has incentive to change to an edge on the first level that is used by some player in $\{j,\dots,i-1\}$. (i has higher prio) ⇒ No Nash.