Weighted Congestion Games

Def:
- player set $\mathcal{N} = \{1, \dots, n\}$, weights $w_i > 0 \ \forall i \in \mathcal{N}$
- finite set of resources R
- strategy spaces $\Sigma_i \subseteq 2^R \ \forall i \in \mathcal{N}$
- cost function $d_r : \mathbb{R} \to \mathbb{R} \ \forall r \in R$

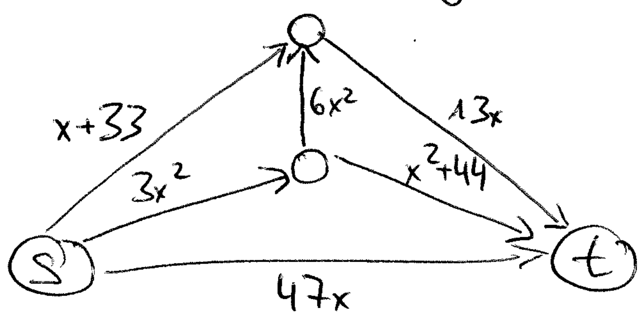
Cost of player i under S:

$$c_i(S) = \sum_{r \in S_i} w_i \cdot d_r(n_r(S)) \quad \text{with } n_r(S) = \sum_{i : r \in S_i} w_i$$

Theorem (Goemans, Mirrokni, Vetta (2005)) There is a weighted congestion game without pure Nash equilibrium.

Proof: $\mathcal{N} = \{1, 2\}$, $w_1 = 1$, $w_2 = 2$,

symmetric network congestion game



(d_r depicted next the edges)

There are 4 strategies per player. Checking all states yields the result. □

<u>Theorem</u> (Fotakis, Kontogiannis, Spirakis, 2005)

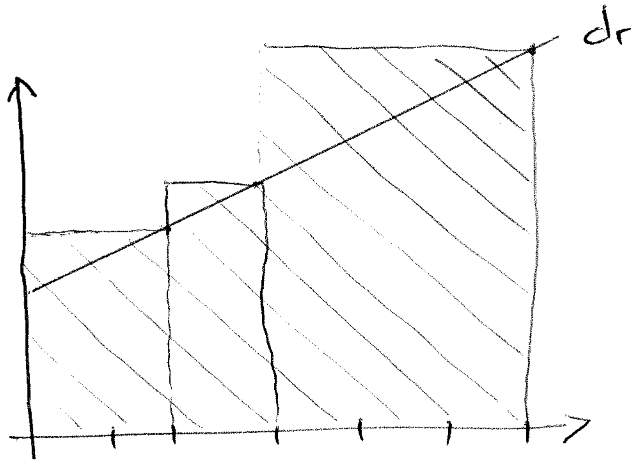
Weighted congestion games with affine cost functions have a pure Nash equilibrium.

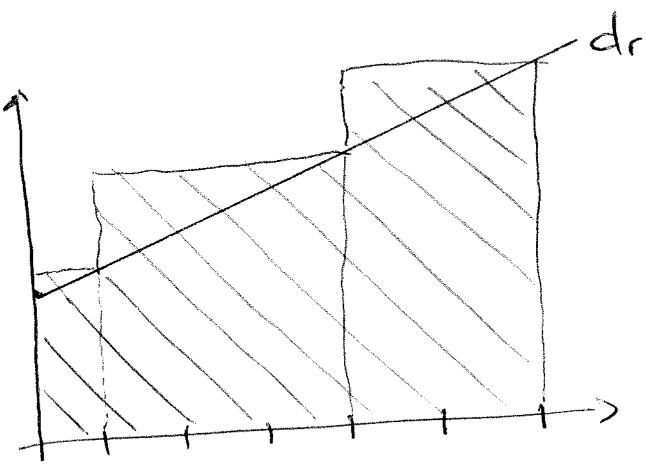
<u>Proof:</u> Consider the function $\Phi: \Sigma \to \mathbb{R}$ defined by

$$\Phi(S) = \sum_{r\in R} \sum_{i\in N: r\in S_i} w_i \, d_r\Big(\sum_{k\in\{1,\dots,i\}: r\in S_k} w_k \Big)$$

(Note: For $w_i = 1$ $\forall i \in N$ this is equal to Rosenthal's potential function.)

The crucial observation is that for affine functions, the function Φ is independent of the ordering of the players:





more formally:

Let $d_r(x) = a_r x + b_r$.

$$\Rightarrow \Phi(S) = \sum_{r\in R} \sum_{i\in N: r\in S_i} w_i d_r\Big(\sum_{j\in\{1,\dots,i\}: r\in S_j} w_j \Big)$$

$$= \sum_{r\in R} \Big(b_r \cdot n_r(S) + \sum_{i,j\in N: r\in S_i\cap S_j, j\le i} a_r w_i w_j \Big)$$

$$= \sum_{r\in R} \Big(b_r n_r(S) + \frac{1}{2} \sum_{i,j\in N: r\in S_i\cap S_j} a_r w_i w_j + \frac{1}{2} \sum_{i\in N: r\in S_i} a_r w_i^2 \Big)$$

$\Rightarrow \Phi(S)$ independent on the ordering.

Need to show: $\Phi(S_i', S_{-i}) - \Phi(S) = C_i(S_i', S_{-i}) - C_i(S) \quad \forall i\in N$.

Since Φ is indep. on the ordering, assume $i = n$.

$$\Phi(S_n', S_{-n}) = \Phi(S) + w_n \sum_{r\in S_n'\setminus S_n} d_r(n_r(S_n', S_{-n}))$$

$$- w_n \sum_{r\in S_n\setminus S_n'} d_r(n_r(S))$$

$$= \Phi(S) + C_n(S_n', S_{-n}) - C_n(S)$$

$\Rightarrow \Phi$ is an exact potential function. $\square$.

<u>Def.</u> A set $\mathcal{C}$ of cost functions $d: \mathbb{R}_{\ge 0} \to \mathbb{R}$ is called consistent (for weighted congestion games) if every finite weighted congestion game with the property that $d_r \in \mathcal{C}$ for all $r\in R$ has a PNE.

Theorem (Harks, Klimm 2010)

A set $\mathcal{C}$ of strictly increasing and continuous functions is consistent for weighted congestion games if and only if one of the following holds:

1. $\mathcal{C}$ contains only affine functions (of the form $d_r(x) = a_r x + b_r$)

2. $\mathcal{C}$ contains only exponential functions of the form $d_r(x) = a_r e^{\lambda x} + b_r$, where λ is equal for all functions in $\mathcal{C}$.

(without proof)