Algorithmic Game Theory

Winter Term 2019 / 2020

Prof. Dr. Martin Hoefer, Dr. Daniel Schmand

JOHANN WOLFGANG GOETHE UNIVERSITÄT FRANKFURT AM MAIN

Algorithmen und Komplexität
Institut für Informatik

Exercise Sheet 9

Publication: Jan 14, 2019
Solutions Due: Jan 21, 2019

Please hand in your solutions until Tuesday, January 21, 10:15h, in H9 or in the letterbox between rooms 114 and 115, R.M.S. 11-15.

Exercise 9.1. (3+3 Points)

a) We stated in the lecture that the FPTAS with granularity parameter $s = \varepsilon \cdot v_{\max}/n$ is not necessarily monotone (see slides, page 56). Prove this statement, i.e. give an example, where the allocation defined by the FPTAS is not monotone for at least one bidder.

b) Prove that the same scheme with $s = \delta$ independent on $v_1, \ldots, v_n$ is monotone for all bidders.

Exercise 9.2. (3+3 Points)

Consider an auction with k identical items. Each bidder is only interested in getting one of the items. The seller wants to get at least a total revenue of $R \geq 0$. The mechanism is defined as follows.

Collect the bids $b_1, \ldots, b_n$.
Initialize the set S with the k highest bidders, set payments $p_i = 0$ for all $i \in [n]$.
while *there is a bidder* $i \in S$ *with* $b_i < R/|S|$ **do**
    Delete such a bidder from S.
if $S \neq \emptyset$ **then**
    for *all bidders* $i \in S$ **do**
        allocate one item to player i and set $p_i = R/|S|$.

---

a) Show the following: Let M be any normalized, and incentive compatible mechanism with the property that all players that get an item have to pay the same. If M generates a revenue of at least R, than the mechanism above also generates a revenue of at least R.

b) Show the following: There is an example for an incentive compatible mechanism M that guarantees a revenue of at least R, where the mechanism above does not generate a revenue of R. (In this case, M does not have the property that all players that get an item have to pay the same.)

Exercise 9.3. (2+2+2 Points)

This exercise derives an interesting interpretation of a virtual valuation $\varphi(v) = v - \frac{1-F(v)}{f(v)}$ and the regularity condition. Consider a strictly increasing distribution function F with a strictly positive density f on the interval $[0, v_{max}]$ (with $v_{max} < \infty$). For $q \in [0,1]$, define $V(q) = F^{-1}(1-q)$ as the posted price resulting in a probability q of a sale (for a single bidder with valuation drawn from F). Define $R(q) = q \cdot V(q)$ as the expected revenue obtained when (for a single bidder) the probability of a sale is q. The function $R(q)$, for $q \in [0,1]$, is often called the *revenue curve* of a distribution F. Note that $R(0) = R(1) = 0$.

a) What is the revenue curve for the uniform distribution on [0,1]?

b) Prove that for general F the slope of the revenue curve at q (i.e.,$R'(q)$) is precisely $\varphi(V(q))$, where φ is the virtual valuation function for F. (*Hint: There is a well-known formula for obtaining a derivative of an inverse function. You can use it here.*)

c) Prove that a distribution is regular if and only if its revenue curve is concave.

Exercise 9.4. (3+3 Points)

a) Consider the VCG-auction with k identical items, for the case that every bidder only wants to get one item. (Allocate k items to the k highest bidders, everyone of these pays the $(k+1)$-highest bid). Can you extend the theorem of Bulow-Klemperer to this case? In case it works, how many additional bidders are needed, such that the VCG-auction with $n + x$ bidders guarantees at least the same expected revenue as the one with the optimal mechanism with n bidders?

b) Consider a one-item auction with n bidders. For each bidder, we draw a valuation v_i independently from a common distribution $\mathcal{V}$ with non-negative support. Now, consider the payments p of a Vickrey-second-price-auction (without reservation price) and payments p^* of an optimal mechanism (with the same n bidders). Show the following statement:

$$\mathbb{E}_{v\sim\mathcal{V}^n}\left[\sum_{i=1}^{n} p_i(v)\right] \quad \geq \quad \left(\frac{n-1}{n+1}\right) \cdot \mathbb{E}_{v\sim\mathcal{V}^n}\left[\sum_{i=1}^{n} p_i^*(v)\right] \quad .$$