

# Strategic Games and Nash Equilibrium

Algorithmic Game Theory

Winter 2022/23

Normal Form Games

PPAD

Zero-Sum Games

Appendix: LP Duality

# Prisoner's Dilemma

|   | S    | C    |
|---|------|------|
| S | 2, 2 | 5, 1 |
| C | 1, 5 | 4, 4 |

- ▶ Two criminals interrogated separately.
- ▶ Strategies: (C)onfess, remain (S)ilent
- ▶ Confessing yields a smaller verdict if the other one is silent
- ▶ If both confess, the verdict is larger for both (4 years) compared to when they both remain silent (2 years).

# Prisoner's Dilemma

|   | S    | C    |
|---|------|------|
| S | 2, 2 | 5, 1 |
| C | 1, 5 | 4, 4 |

- ▶ If both players remain (S)ilent, the total cost is smallest.
- ▶ If both players (C)onfess, the cost is larger for both of them.
- ▶ Still, for each player confessing is always the preference!

# Normal Form Games

## Definition

A normal form game is a triple  $(\mathcal{N}, (S_i)_{i \in N}, (c_i)_{i \in N})$  where

- ▶  $\mathcal{N}$  is the set of **players**,  $n = |\mathcal{N}|$ ,
- ▶  $S_i$  is the set of **(pure) strategies** of player  $i$ ,
- ▶  $S = S_1 \times \dots \times S_n$  is the set of states,
- ▶ a **state** is  $s = (s_1, \dots, s_n) \in S$ ,
- ▶  $c_i : S \rightarrow \mathbb{R}$  is the **cost function** of player  $i \in \mathcal{N}$ . In state  $s$  player  $i$  has a cost of  $c_i(s)$ .

We denote by  $s_{-i} = (s_1, \dots, s_{i-1}, s_{i+1}, \dots, s_n)$  a state  $s$  without the strategy  $s_i$ .

# Dominant Strategies

## Definition

A pure strategy  $s_i$  is called a **dominant strategy** for player  $i \in \mathcal{N}$  if  $c_i(s_i, s_{-i}) \geq c_i(s'_i, s_{-i})$  for every  $s'_i \in S_i$  and every  $s_{-i}$ .

## Definition

A state  $s = (s_1, \dots, s_n)$  is called a **dominant strategy equilibrium** if for every player  $1 \leq i \leq n$  strategy  $s_i \in S_i$  is a dominant strategy.

Does every game have a dominant strategy equilibrium? No!

# Pareto Optimum

## Definition

A state  $s$  **Pareto-dominates** another state  $s'$  (or:  $s$  is a **Pareto improvement** over  $s'$ ) if  $c_i(s) \leq c_i(s')$  for every player  $i \in \mathcal{N}$  and  $c_j(s) < c_j(s')$  for at least one player  $j \in \mathcal{N}$ .

## Definition

A state  $s$  is called a **Pareto optimum** or **Pareto efficient** if there is no state that Pareto dominates  $s$ .

In a Pareto optimum a player might be able to strictly decrease its cost by deviating – however, no player can strictly decrease its cost without strictly increasing the cost of another player.

Does every game have a Pareto optimum? Yes!

## Battle of the Sexes

|             | (Z)eil | (E)intracht |
|-------------|--------|-------------|
| (Z)eil      | 1      | 6           |
| (E)intracht | 5      | 2           |

- ▶ In state (Z,Z) the preference for both is (Z)eil.
  - ▶ In state (E,E) the preference for both is (E)intracht.
- ⇒ No global preference.

What is a plausible outcome in this situation?



# Pure Nash Equilibrium

## Definition

A strategy  $s_i$  is called a **best response** against a collection of strategies  $s_{-i}$  if  $c_i(s_i, s_{-i}) \leq c_i(s'_i, s_{-i})$  for all  $s'_i \in S_i$ .

Note:  $s_i$  dominant strategy  $\Leftrightarrow s_i$  best response for all  $s_{-i}$ .

## Definition

A state  $s = (s_1, \dots, s_n)$  is called a **pure Nash equilibrium** if  $s_i$  is a best response against the other strategies  $s_{-i}$  for every player  $1 \leq i \leq n$ .

A Nash equilibrium

- ▶ ... is a collection of local preferences in the game.
- ▶ ... is stable against unilateral deviation.

Does every game have a pure Nash equilibrium? No!

# Rock-Paper-Scissors



|   | R  | P  | S  |
|---|----|----|----|
| R | 0  | -1 | 1  |
| P | 1  | 0  | -1 |
| S | -1 | 1  | 0  |

# Rock-Paper-Scissors



|   | R       | P       | S       |
|---|---------|---------|---------|
| R | 0<br>↓→ | -1<br>↓ | 1<br>←  |
| P | 1<br>→  | 0<br>↓→ | -1<br>↑ |
| S | -1<br>↑ | 1<br>←  | 0<br>←↑ |

# Mixed Strategies

## Definition

A **mixed strategy**  $x_i$  for player  $i$  is a probability distribution over the set of pure strategies  $S_i$ .

For finite games  $x_i$  is such that  $x_{ij} \in [0, 1]$  and  $\sum_{j \in S_i} x_{ij} = 1$ .

The cost of a mixed state for player  $i$  is

$$c_i(x) = \sum_{s \in S} p(s) \cdot c_i(s) ,$$

where

$$p(s) = \prod_{i \in \mathcal{N}, j = s_i} x_{ij}$$

is the probability that the outcome is pure state  $s$ .

# Mixed Nash Equilibrium

## Definition

A **(mixed) best response strategy**  $x_i$  against a collection of mixed strategies  $x_{-i}$  is such that  $c(x_i, x_{-i}) \leq c_i(x'_i, x_{-i})$  for all other mixed strategies  $x'_i$ .

## Definition

A mixed state  $x$  is called a **(mixed) Nash equilibrium** if  $x_i$  is a best response strategy against  $x_{-i}$  for every player  $1 \leq i \leq n$ .

Note:

- ▶ Every pure strategy is also a mixed strategy.
- ▶ Every pure Nash equilibrium is also a mixed Nash equilibrium.

## Example

|     |   |   |   |
|-----|---|---|---|
|     | 0.3   | 0.7   |   |
| 0.2 | 2   | 3   | $0.3 \cdot 1 + 0.7 \cdot 2$<br>$= 0.3 + 1.4$<br>$= 1.7$ |
| 0.8 | 4   | 2   | $0.3 \cdot 1 + 0.7 \cdot 5$<br>$= 0.3 + 3.5$<br>$= 3.8$ |
|     | $0.2 \cdot 2 + 0.8 \cdot 4$<br>$= 0.4 + 3.2$<br>$= 3.6$ | $0.2 \cdot 3 + 0.8 \cdot 2$<br>$= 0.6 + 1.6$<br>$= 2.2$ |   |

- ▶  $c_1(x) = 1.7 \cdot 0.2 + 3.8 \cdot 0.8 > 1.7$  – best response is (1, 0)
- ▶  $c_2(x) = 3.6 \cdot 0.3 + 2.2 \cdot 0.7 > 2.2$  – best response is (0, 1)
- ▶ State  $x$  with  $x_1 = (0.2, 0.8)$  and  $x_2 = (0.3, 0.7)$  is no mixed Nash equilibrium.

# Observation

In the previous example  $x$  is not a mixed Nash equilibrium, because players play suboptimal pure strategies with positive probability.

## Fact

If a mixed best response  $x_i$  against  $x_{-i}$  has  $x_{ij} > 0$ , then  $j$  is a pure best response against  $x_{-i}$ .

The cost of  $x_i$  is a “weighted average” of the cost of the pure strategies. It is minimal if and only if the averaging is just over pure strategies with minimum cost.

# Example

|   |                                  |                                  |                                  |
|---|----------------------------------|----------------------------------|----------------------------------|
|   | 1                                | 0                                |                                  |
| 1 | 2                                | 3                                | $1 \cdot 1 + 0 \cdot 2$<br>$= 1$ |
| 0 | 4                                | 2                                | $1 \cdot 1 + 0 \cdot 5$<br>$= 1$ |
|   | $1 \cdot 2 + 0 \cdot 4$<br>$= 2$ | $1 \cdot 3 + 0 \cdot 2$<br>$= 3$ |                                  |

- ▶ State  $x$  with  $x_1 = (1, 0)$  and  $x_2 = (1, 0)$  is a pure (and hence also a mixed) Nash equilibrium.



## Example

|               | 1  | 0  |                                  |
|---------------|--|--|----------------------------------|
| $\frac{2}{3}$ | 2  | 3  | $1 \cdot 1 + 0 \cdot 2$<br>$= 1$ |
| $\frac{1}{3}$ | 4  | 2  | $1 \cdot 1 + 0 \cdot 5$<br>$= 1$ |
|               | $\frac{2}{3} \cdot 2 + \frac{1}{3} \cdot 4$<br>$= \frac{8}{3}$ | $\frac{2}{3} \cdot 3 + \frac{1}{3} \cdot 2$<br>$= \frac{8}{3}$ |                                  |

- ▶ State  $x$  with  $x_1 = (\frac{2}{3}, \frac{1}{3})$  and  $x_2 = (1, 0)$  is a mixed Nash equilibrium.
- ▶ For the row player the upper strategy is a dominant strategy, but in the first column it is not *strictly* better. If it was strictly better in every column, the lower strategy would not be played in any mixed Nash equilibrium. (Why?)

# Nash Theorem

## Theorem (Nash Theorem)

*Every finite normal form game has a mixed Nash equilibrium.*

We will use Brouwer's fixed point theorem to prove it.

## Theorem (Brouwer Fixed Point Theorem)

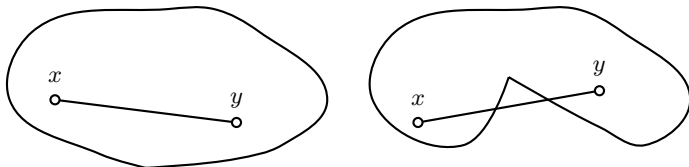
*Every continuous function  $f : D \rightarrow D$  mapping a compact and convex nonempty subset  $D \subseteq \mathbb{R}^m$  to itself has a fixed point  $x^* \in D$  with  $f(x^*) = x^*$ .*

# Brouwer's Theorem: Prerequisites and Definitions

- ▶ A set  $D \subset \mathbb{R}^m$  is **convex** if for any  $x, y \in D$  and any  $\lambda \in [0, 1]$  we have  $\lambda x + (1 - \lambda)y \in D$ .
- ▶ A subset  $D \subset \mathbb{R}^m$  is **compact** if and only if it is closed and bounded.
- ▶ A set  $D \subseteq \mathbb{R}^m$  is **bounded** if and only if there is some integer  $M \geq 0$  such that  $D \subseteq [-M, M]^m$ .
- ▶ Consider a set  $D \subseteq \mathbb{R}^m$  and a sequence  $x_0, x_1, \dots$ , where for all  $i \geq 0$ ,  $x_i \in D$  and there is  $x \in \mathbb{R}^m$  such that  $x = \lim_{i \rightarrow \infty} x_i$  (i.e., for all  $\epsilon > 0$  there is integer  $k > 0$  such that  $\|x - x_j\|_2 < \epsilon$  for all  $j > k$ ). A set  $D$  is **closed** if  $x \in D$  for every such sequence.
- ▶ A function  $f : D \rightarrow \mathbb{R}^m$  is continuous at a point  $x \in D$  if for all  $\epsilon > 0$ , there exists  $\delta > 0$ , such that for all  $y \in D$ : If  $\|x - y\|_2 < \delta$  then  $\|f(x) - f(y)\|_2 < \epsilon$ .  $f$  is called **continuous** if it is continuous at every point  $x \in D$ .

# Brouwer's Theorem: Prerequisites and Examples

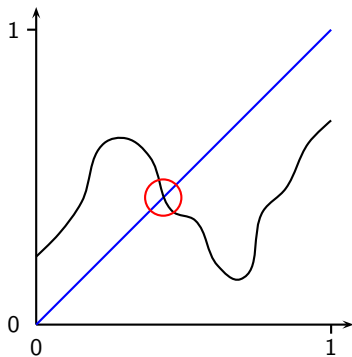
- ▶ Convex/Non-convex:



- ▶ Closed and bounded:
  - $[0, 1]^2$  is closed and bounded.
  - $[0, 1)$  is not closed but bounded.
  - $[0, \infty)$  is closed and unbounded.
- ▶ Continuous: Clear.

# Brouwer's Theorem: Example

Every continuous  $f : [0, 1] \rightarrow [0, 1]$  has a fixed point:



For  $f : [0, 1]^2 \rightarrow [0, 1]^2$ : Crumpled Sheet Experiment

# Nash Theorem

## Theorem (Nash Theorem)

*Every finite normal form game has a mixed Nash equilibrium.*

### Proof:

First check the conditions of Brouwer's Theorem.

### Fact

*The set  $X$  of mixed states  $x = (x_1, \dots, x_n)$  of a finite normal form game is a convex compact subset of  $\mathbb{R}^m$  with  $m = \sum_{i=1}^n m_i$  with  $m_i = |S_i|$ .*

We will define a function  $f : X \rightarrow X$  that transforms a state into another state. The fixed points of  $f$  are shown to be the mixed Nash equilibria of the game.

# Properties of Nash Equilibria

Recall:

- ▶ A mixed Nash equilibrium  $x$  is a collection of mixed best responses  $x_i$ .
- ▶ If a best response  $x_i$  against  $x_{-i}$  has  $x_{ij} > 0$ , then  $j \in S_i$  is pure best response against  $x_{-i}$ .
- ▶ A collection of best responses (i.e. a mixed Nash equilibrium)  $x = (x_1, \dots, x_n)$  has

$$c_i(x) - c_i(j, x_{-i}) \leq 0 \quad \text{for all } j \in S_i \text{ and all } i \in \mathcal{N}$$

# Proof of Nash's Theorem: Definition

- ▶ For mixed state  $x$  let

$$\phi_{ij}(x) = \max\{0, c_i(x) - c_i(j, x_{-i})\} .$$

- ▶ Define  $f : X \rightarrow X$  with  $f(x) = x' = (x'_1, \dots, x'_n)$  by

$$x'_{ij} = \frac{x_{ij} + \phi_{ij}(x)}{1 + \sum_{k=1}^{m_i} \phi_{ik}(x)}$$

for all  $i = 1, \dots, n$  and  $j = 1, \dots, m_i$ .

## Fact

*f satisfies the prerequisites of Brouwer's Theorem: f is continuous and if  $x \in X$ , then  $f(x) = x' \in X$  is a mixed state.*

(Check as an exercise.)



# Example

- ▶ Player  $i$  has 3 pure strategies
- ▶ Current mixed strategy  $x_i = (0.2, 0.5, 0.3)$
- ▶ Current costs for strategies  $c_i(\cdot, x_{-i}) = (2.2, 4.2, 2.2)$
- ▶ Current cost  $c(x_i, x_{-i}) = 3.2$
- ▶ Under these conditions strategy  $x_i$  is mapped to  $x'_i$  as follows:

| $x_{ij}$ | $c_i(j, x_{-i})$ | $\phi_{ij}(x)$ | $x'_{ij}$                         |
|----------|------------------|----------------|-----------------------------------|
| 0.2      | 2.2              | 1              | $\frac{0.2+1}{1+2} = 0.4$         |
| 0.5      | 4.2              | 0              | $\frac{0.5+0}{1+2} \approx 0.166$ |
| 0.3      | 2.2              | 1              | $\frac{0.3+1}{1+2} \approx 0.434$ |

# Fixed Points

Brouwers Theorem tells us that there is  $x^*$  with  $f(x^*) = x^*$ . We need to show two directions:

$$f(x) = x \quad \Leftrightarrow \quad x \text{ is mixed Nash equilibrium.}$$

Easy:  $x$  is mixed Nash  $\Rightarrow f(x) = x$ : All  $\phi_{ij}(x) = 0$ .

To show:  $x^* = f(x^*) \Rightarrow x^*$  is a mixed Nash equilibrium.

# Fixed Points as Nash Equilibria

For each  $i = 1, \dots, n$  and  $j = 1, \dots, m_i$  we have

$$x_{ij}^* = \frac{x_{ij}^* + \phi_{ij}(x^*)}{1 + \sum_{k=1}^{m_i} \phi_{ik}(x^*)} ,$$

so

$$x_{ij}^* \cdot \left( 1 + \sum_{k=1}^{m_i} \phi_{ik}(x^*) \right) = x_{ij}^* + \phi_{ij}(x^*) ,$$

and

$$x_{ij}^* \sum_{k=1}^{m_i} \phi_{ik}(x^*) = \phi_{ij}(x^*) .$$

We will show that  $\sum_{k=1}^{m_i} \phi_{ik}(x^*) = 0$ . This means that  $x_i^*$  chooses only pure best responses and implies that it is a mixed best response.

# Fixed Points as Nash Equilibria

## Claim

*For every mixed state  $x$  and every player  $i \in \mathcal{N}$ , there is some pure strategy  $j \in S_i$  such that  $x_{ij} > 0$  and  $\phi_{ij}(x) = 0$ .*

## Proof of Claim:

Note that  $c_i(x) = \sum_{j=1}^{m_i} x_{ij} \cdot c_i(j, x_{-i})$ , so there must be some  $j$  with  $x_{ij} > 0$  and cost no less than this “weighted average”.

More formally, there is  $j$  with  $x_{ij} > 0$  and

$$c_i(x) - c_i(j, x_{-i}) \leq 0 .$$

Therefore,  $\phi_{ij}(x) = \max\{0, c_i(x) - c_i(j, x_{-i})\} = 0$ . □

# Fixed Points as Nash Equilibria

For every player  $i$  we consider strategy  $j$  from the claim. This implies  $x_{ij}^* > 0$  and

$$x_{ij}^* \cdot \sum_{k=1}^{m_i} \phi_{ik}(x^*) = \phi_{ij}(x^*) = 0 .$$

Since  $x_{ij}^* > 0$  it must hold that

$$\sum_{k=1}^{m_i} \phi_{ik}(x^*) = 0 ,$$

so  $\phi_{ik}(x^*) = 0$  for all  $k = 1, \dots, m_i$ . Therefore

$$c_i(x^*) \leq c_i(j, x_{-i}^*) \quad \text{for all } j \in S_i.$$

Hence,  $x_i^*$  is a best response. This proves Nash's Theorem. □

Normal Form Games

PPAD

Zero-Sum Games

Appendix: LP Duality

# Computing Nash Equilibria

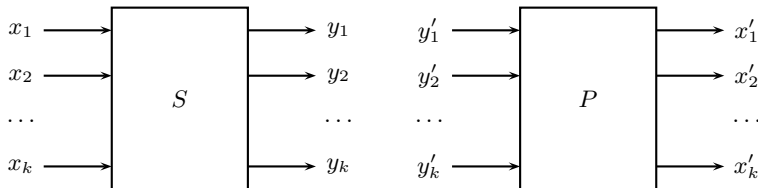
How can we compute a mixed Nash equilibrium?  
What is the **complexity** of computing a Nash equilibrium?

This problem is different from problems we usually encounter:

- ▶ No optimization, trivial as decision problem (existence guaranteed)
- ▶ Search problem, **find** Nash equilibrium.
- ▶ Different Complexity Class: PPAD  
(polynomial parity argument, directed case)
- ▶ A notion of **completeness**, similar to NP:  
Define PPAD-complete problem, construct polynomial-time reductions

There are 3-player games with rational payoff, in which all mixed Nash equilibria have irrational probability values. Thus, we can only hope to obtain **approximations to mixed Nash equilibria or Brouwer fixed points**.

# A PPAD-Complete Problem



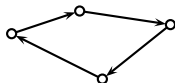
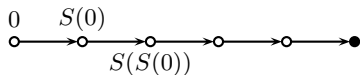
An instance of the END-OF-LINE search problem is given by

- ▶ Two circuits  $S$  and  $P$ , same number of inputs and output bits
- ▶  $S$  and  $P$  define a directed graph:
  - Vertices:  $k$ -bit vectors
  - Edges: There is a directed edge  $(x, y)$  if  $S(x) = y$  and  $P(y) = x$
- ▶  $S$  and  $P$  are such that the **all-0-vector** has one outgoing edge and no incoming edge!

Problem: Find a different source or sink in the graph.



# END-OF-LINE



●  $\equiv$  possible solution

Observations:

- ▶ Every vertex in the graph has indegree and outdegree at most 1.
- ▶ By parity argument END-OF-LINE always admits a solution.
- ▶ Not necessarily the end of the line from 0, finding this specific sink is PSPACE-complete.
- ▶ Only **circuits are the input!** The graph is exponentially large in the input size. It cannot be fully enumerated in polynomial time.

Computing a solution to END-OF-LINE is PPAD-complete.

It is believed that there is no efficient algorithm for this problem.

# Finding (Approximate) Brouwer Fixed Points

## Lemma

*Finding an (approximate) mixed Nash equilibrium is in PPAD.*

### Proof Sketch:

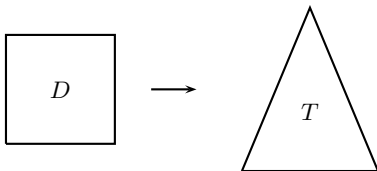
- ▶ Reduction: Finding fixed points with END-OF-LINE
- ▶ Subdivide the space into finite number of smaller areas
- ▶ Find an area close to a fixed point (Approximation)
- ▶ By continuity: Finer granularity yields more precise approximation.

Divide the space into simplices (“multidimensional triangles”) and color vertices according to direction of Brouwer function

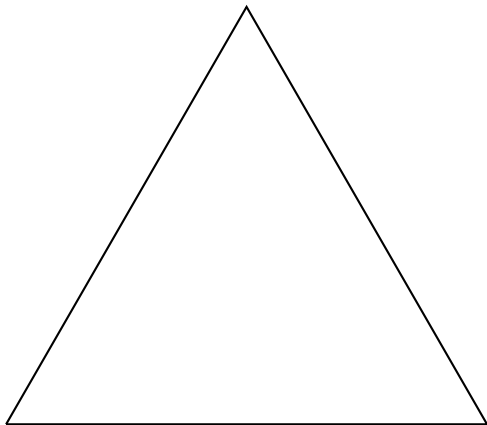
For simplicity of presentation we here consider only problems with  $D \subseteq \mathbb{R}^2$ , e.g.,  $f : [0, 1]^2 \rightarrow [0, 1]^2$ .

# Triangles

For simplicity we transform representation of  $[0, 1]^2$  to a triangle  $T$ . Equivalent fixed point problem with  $f' : T \rightarrow T$ .

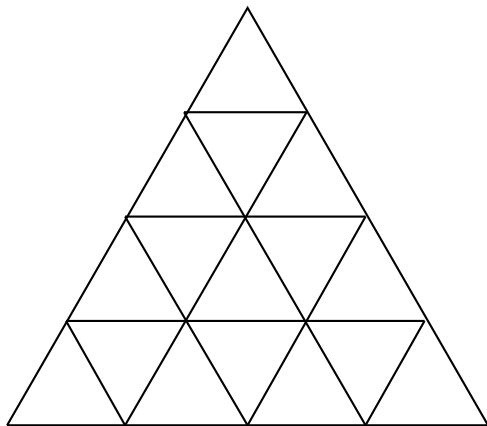


# Subdivision and Coloring



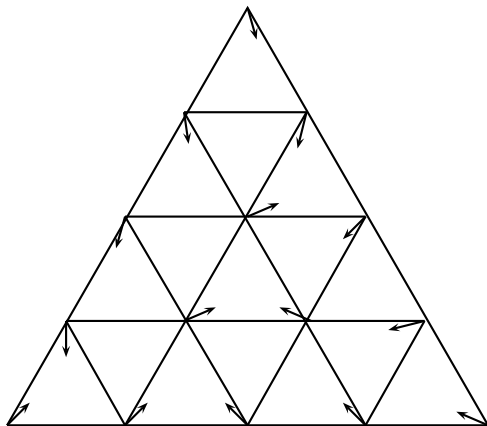
- ▶ The triangle space  $T$  is subdivided into smaller triangles

# Subdivision and Coloring



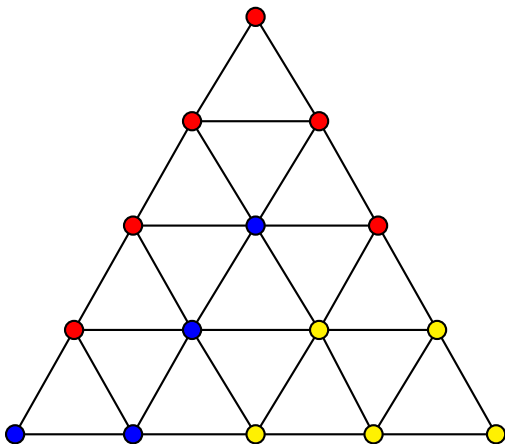
- ▶ The triangle space  $T$  is subdivided into smaller triangles
- ▶ For each vertex consider the direction, in which  $f'$  maps the point

# Subdivision and Coloring



- ▶ The triangle space  $T$  is subdivided into smaller triangles
- ▶ For each vertex consider the direction, in which  $f'$  maps the point
- ▶ Depending on the direction the vertex receives a color.

# Subdivision and Coloring



- ▶ The triangle space  $T$  is subdivided into smaller triangles
- ▶ For each vertex consider the direction, in which  $f'$  maps the point
- ▶ Depending on the direction the vertex receives a color.
- ▶ With increasing granularity trichromatic triangles become the fixed points of  $f'$ .

# Sperner Coloring

## Definition

A **subdivided triangle** is a division of a triangle into smaller triangles.

## Definition

A **Sperner coloring** of the vertices of a subdivided triangle satisfies:

- ▶ Each extremal vertex gets a different color.
- ▶ A vertex on a side of the largest triangle gets a color of one of the corresponding endpoints.
- ▶ Other vertices are colored arbitrarily.

Verify that our coloring based on directions of  $f'$  yields a Sperner coloring.



# Sperner's Lemma

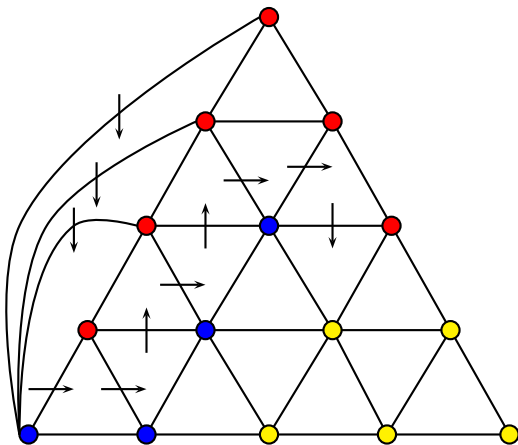
## Lemma (Sperner's Lemma)

*Every Sperner coloring of a subdivided triangle contains a trichromatic triangle.*

### Proof:

- ▶ Connect all vertices on the outer blue/red edge to the blue vertex. Start at the outside face and move over lines connecting a red and a blue vertex. There are at most 2 such lines in each triangle, never visit a triangle twice.
- ▶ This implies an instance of END-OF-LINE:  
Vertices: Small triangles  
Edges: There is an edge if two triangles share a line between a red and blue vertex.
- ▶ By construction indegree and outdegree at most 1
- ▶ There is a starting point by creation, other sources/sinks are the trichromatic triangles. □

## Proof by END-OF-LINE



## Implications and Results

Sperner's Lemma is a *discretized version of Brouwer's fixed point theorem*. The proof of the lemma...

- ▶ shows that Sperner colorings create an instance of END-OF-LINE.
- ▶ can be generalized to more dimensions and simplices instead of triangles. Then trichromatic triangles correspond to simplices with maximum number of colors.
- ▶ with “infinite granularity” implies maximally colored simplices are Brouwer fixed points.

This proves that finding a Brouwer fixed point and, hence, a mixed Nash equilibrium in a finite game is in PPAD. □

Fundamental result in the literature by Daskalakis/Papadimitriou and Chen/Deng/Teng:

### Theorem

*Finding a mixed Nash equilibrium in a finite 2-player game is PPAD-complete.*

Normal Form Games

PPAD

Zero-Sum Games

Appendix: LP Duality

## Definition

For consistency with literature we here consider utility functions instead of cost.

### Definition

The **utility** of player  $i$  in a state  $s$  of a normal form game is  $u_i(s) = -c_i(s)$ .

### Definition

A **zero-sum game** is a strategic game, in which for every state  $s$  we have  $\sum_{i \in \mathcal{N}} u_i(s) = 0$ .

In a zero-sum game every utility gain of one player results in a utility loss of another player. For instance, this can be used to model situations in which players must divide a common good.

## 2-player Zero Sum Games

Two players, player I (row player), player II (column player)

Representation as a matrix  $A \in \mathbb{R}^{k \times \ell}$  with  $k = |S_I|$  rows and  $\ell = |S_{II}|$  columns:

$$\begin{pmatrix} a_{11} & a_{12} & \cdot & \cdot & \cdot & a_{1\ell} \\ a_{21} & a_{22} & \cdot & \cdot & \cdot & a_{2\ell} \\ \cdot & \cdot & \cdot & & & \cdot \\ \cdot & \cdot & & \cdot & & \cdot \\ \cdot & \cdot & & & \cdot & \cdot \\ a_{k1} & a_{k2} & \cdot & \cdot & \cdot & a_{k\ell} \end{pmatrix}$$

$a_{ij}$  is utility for player I

$-a_{ij}$  is utility for player II

## Examples

Matching Pennies

$$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

Rock-Paper-Scissors

$$\begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}$$

A game with  $k \neq \ell$ :

$$\begin{pmatrix} 0 & 2 & 4 \\ 1 & 2 & 3 \end{pmatrix}$$

# Utility by Matrix Multiplication

We denote mixed strategies by  $x$  for I and  $y$  for II.

Computing the utility  $u_I(x, y)$ :

$$\begin{array}{c|ccc} & 0.1 & 0.4 & 0.5 \\ \hline 0.8 & 0 & 2 & 4 \\ 0.2 & 1 & 2 & 3 \end{array} \longrightarrow \begin{aligned} & 0.8 \cdot (0.1 \cdot 0 + 0.4 \cdot 2 + 0.5 \cdot 4) \\ & + 0.2 \cdot (0.1 \cdot 1 + 0.4 \cdot 2 + 0.5 \cdot 3) \\ & = 2.48 \end{aligned}$$

$$\begin{aligned} u_I(x, y) = -u_{II}(x, y) &= \sum_{i=1}^k \sum_{j=1}^{\ell} x_i a_{ij} y_j \\ &= \begin{pmatrix} x_1 & x_2 \end{pmatrix} \cdot \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} \cdot \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \\ &= x^T A y \end{aligned}$$



# Public Strategy Choices

Suppose player I has to decide first. He must pick a public strategy before player II makes his choice. How should I choose his public strategy?

$$\begin{pmatrix} 0 & 2 & 4 \\ 1 & 2 & 3 \end{pmatrix}$$

Player II will hurt player I as much as possible.

In this game II will always answer with column 1. Hence, optimal choice for I is pure strategy 2 or  $x = (0, 1)^T$ .

# Maximin Strategies

Player I picks  $x$ , then player II best responds with  $y$ . II solves the problem  $\max_y u_{II}(x, y) = \max_y -x^T Ay = \min_y x^T Ay$ .

Hence, player I searches for  $x$  that maximizes  $\min_y x^T Ay$ .

## Definition

The **gain-floor** of a 2-player zero-sum game is

$$v_I^* = \max_x \min_y x^T Ay .$$

A strategy  $x^*$  that yields the gain-floor is an **optimal strategy** for I, called **maximin strategy**.

## Example Maximin

$$\begin{pmatrix} 5 & 1 & 2 \\ 1 & 4 & 3 \end{pmatrix}$$

Player II will hurt player I as much as possible.

- ▶ I picks row 1  $\Rightarrow$  II picks column 2  $\Rightarrow$  I gets utility 1
- ▶ I picks row 2  $\Rightarrow$  II picks column 1  $\Rightarrow$  I gets utility 1
- ▶ I picks  $x = (0.5, 0.5)$ , minimum loss for II is 2.5 in columns 2 and 3  
 $\Rightarrow$  I gets utility 2.5!

What is  $x^*$ , how large can  $v_I^*$  be?

## Dual Perspective: Minimax

Now suppose player II first picks  $y$ , then player I picks  $x$  optimally with  $\max_x u_I(x, y) = \max_x x^T Ay$ .

Hence, II searches for  $y$  that minimizes  $\max_x x^T Ay$ .

### Definition

The **loss-ceiling** of a 2-player zero-sum game is

$$v_{II}^* = \min_y \max_x x^T Ay .$$

A strategy  $y^*$  that yields the loss-ceiling is an **optimal strategy** for II, called **minimax strategy**.

What is  $y^*$ , how small can  $v_{II}^*$  be?

How do  $v_I^*$  and  $v_{II}^*$  compare?

# Minimax Theorem

Intuitively, if both players play optimally, player I should gain at least  $v_I^*$ , player II should not lose more than  $v_{II}^*$ . It is easy to show that

## Lemma

*It holds that  $v_I^* \leq v_{II}^*$ .*

Perhaps surprisingly, von Neumann and Morgenstern proved

## Theorem (Minimax Theorem)

*In every 2-player zero-sum game it holds that  $v = v_I^* = v_{II}^*$ . The value  $v$  is called the **value** of the game.*

# Minimax Theorem by Linear Programming Duality

Consider the optimization problem to find  $x^*$  and  $v_I^* = \max_x \min_y x^T A y$ .

Observe:

- ▶ II plays a best response  $y$  against  $x$ .
- ▶ For a given  $x$ , player II sets  $y_j > 0$  if and only if his expected loss  $\sum_{i=1}^k x_i a_{ij}$  in column  $j$  is minimal.
- ▶ Hence,

$$v_I = \sum_{j=1}^{\ell} \sum_{i=1}^k x_i a_{ij} y_j = \min_{j=1}^{\ell} \sum_{i=1}^k x_i a_{ij}$$

- ▶ For any  $x$  and the resulting utility  $v_I$  obtained by I we thus know

$$v_I \leq \sum_{i=1}^k x_i a_{ij} \quad \text{for all } j = 1, \dots, \ell.$$

## Gain-Floor Optimization as a Linear Program

$$\begin{array}{ll} \text{Maximize} & v_{\text{I}} \\ \text{subject to} & v_{\text{I}} - \sum_{i=1}^k x_i a_{ij} \leq 0 \quad \text{for all } j = 1, \dots, \ell \\ & \sum_{i=1}^k x_i = 1 \\ & x_i \geq 0 \quad \text{for all } i = 1, \dots, k \\ & v_{\text{I}} \in \mathbb{R} \end{array} \quad (1)$$

# Loss-Ceiling Optimization as a Linear Program

Similar arguments yield a linear program for loss-ceiling minimization.

$$\begin{array}{ll}
 \text{Minimize} & v_{\text{II}} \\
 \text{subject to} & v_{\text{II}} - \sum_{j=1}^{\ell} a_{ij} y_j \geq 0 \quad \text{for all } i = 1, \dots, k \\
 & \sum_{j=1}^{\ell} y_j = 1 \\
 & y_j \geq 0 \quad \text{for all } j = 1, \dots, \ell \\
 & v_{\text{II}} \in \mathbb{R}
 \end{array} \tag{2}$$

In the appendix we show that this represents the LP-dual of the Gain-Floor LP (1).



# Implications

Finding optimal strategies for players I and II can be formulated as dual linear programs.

**Strong duality** in Linear Programming:

- ▶ Consider a linear program with a feasible optimum solution
- ▶ Let  $f^*$  be the optimal objective function value
- ▶ Then the dual has a feasible optimum solution, objective function value  $g^*$
- ▶ Strong Duality: It holds that  $f^* = g^*$ .

Thus, strong duality proves the minimax theorem. □

## Example

$$\begin{pmatrix} 5 & 1 & 2 \\ 1 & 4 & 3 \end{pmatrix}$$

$$\begin{array}{ll} \text{Max.} & v_{\text{I}} \\ \text{s.t.} & v_{\text{I}} - 5x_1 - 1x_2 \leq 0 \\ & v_{\text{I}} - 1x_1 - 4x_2 \leq 0 \\ & v_{\text{I}} - 2x_1 - 3x_2 \leq 0 \\ & x_1 + x_2 = 1 \\ & x_1, x_2 \geq 0 \\ & v_{\text{I}} \in \mathbb{R} \end{array}$$

$$\begin{aligned} x^* &= (0.4, 0.6) \\ v_{\text{I}}^* &= 2.6 \end{aligned}$$

$$\begin{array}{ll} \text{Min.} & v_{\text{II}} \\ \text{s.t.} & v_{\text{II}} - 5y_1 - 1y_2 - 2y_3 \geq 0 \\ & v_{\text{II}} - 1y_1 - 4y_2 - 3y_3 \geq 0 \\ & y_1 + y_2 + y_3 = 1 \\ & y_1, y_2, y_3 \geq 0 \\ & v_{\text{II}} \in \mathbb{R} \end{array}$$

$$\begin{aligned} y^* &= (0.2, 0, 0.8) \\ v_{\text{II}}^* &= 2.6 \end{aligned}$$

Is  $(x^*, y^*)$  a mixed Nash equilibrium?

# Mixed Nash equilibrium

## Corollary

A state  $(x, y)$  in a 2-player zero-sum game is a mixed Nash equilibrium  
 $\Leftrightarrow$   $x$  and  $y$  are optimal strategies for the players.

Proof ( $\Rightarrow$ ):

- ▶ Consider  $(x, y)$  and assume  $x$  is suboptimal (similar for  $y$  suboptimal)
- ▶ There is  $y'$  that achieves  $u_{II}(x, y') > -v$ , thus  $u_I(x, y') < v$ .
- ▶ To be NE we must have  $u_{II}(x, y) \geq u_{II}(x, y')$ , so  $u_I(x, y) < v$ .
- ▶ If  $u_I(x, y) < v$ , I can improve by optimal strategy  $\Rightarrow (x, y)$  no mixed NE.

( $\Leftarrow$ ):

- ▶ Suppose both play optimal, but player I has a better strategy  $x'$
- ▶ This means  $u_I(x', y) > v$ , but then  $y$  is suboptimal for II
- ▶ Same argument for player II having a better strategy.

# Mixed Nash equilibrium

## Corollary

*All mixed Nash equilibria in a 2-player zero-sum game yield an expected utility of  $v$  ( $-v$ ) for player I (II).*

We can find optimal strategies by solving the linear programs (1) and (2). There are efficient algorithms for solving linear programs, which proves the following result:

## Theorem

*In 2-player zero-sum games a mixed Nash equilibrium can be computed in polynomial time.*

# Literature

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Normal Form Games

PPAD

Zero-Sum Games

Appendix: LP Duality

# Constructing the Dual

We construct an upper bound on  $v_I$  for every solution of (1).

- ▶ Consider a solution  $(v_I, x)$  of (1).
- ▶ We take a linear combination of the constraints to construct an upper bound. In particular, we use multipliers  $z_j$  and  $w_I$ :

$$\begin{aligned} z_j \cdot \left( v_I - \sum_{i=1}^k x_i a_{ij} \right) &\leq z_j \cdot 0 \quad \text{for each } j \text{ and} \\ w_I \cdot \sum_{i=1}^k x_i &= w_I \cdot 1 \end{aligned}$$

Here  $z_j \geq 0$  to keep the correct inequality.

# Constructing the Dual

- Now we try to get an upper bound by using the linear combination:

$$\begin{aligned}
 v_I &\leq \sum_{j=1}^{\ell} z_j \left( v_I - \sum_{i=1}^k x_i a_{ij} \right) + w_I \cdot \sum_{i=1}^k x_i \\
 &= \left( \sum_{j=1}^{\ell} z_j \right) \cdot v_I + \sum_{i=1}^k \left( w_I - \sum_{j=1}^{\ell} a_{ij} z_j \right) \cdot x_i \\
 &\leq \sum_{j=1}^{\ell} z_j \cdot 0 + w_I \cdot 1 = w_I
 \end{aligned}$$

- This works if the first inequality is fulfilled, and this holds if the following conditions for coefficients for the  $v_I$  and  $x_i$  on l.h.s. and r.h.s. are true:

$$\begin{aligned}
 1 &= \sum_{j=1}^{\ell} z_j && \text{(Same ones because } v_I \in \mathbb{R}.) \\
 0 &\leq w_I - \sum_{j=1}^{\ell} a_{ij} z_j && \text{(Possibly larger ones, because } x_i \geq 0.)
 \end{aligned}$$

What is the best upper bound  $w_I$  that can be obtained in this way?



## Finding the best Upper Bound

$$\begin{aligned}
 &\text{Minimize} && w_{\text{I}} \\
 &\text{subject to} && w_{\text{I}} - \sum_{j=1}^{\ell} a_{ij} z_j \geq 0 && \text{for all } i = 1, \dots, k \\
 &&& \sum_{j=1}^{\ell} z_j = 1 && (3) \\
 &&& z_j \geq 0 && \text{for all } j = 1, \dots, \ell \\
 &&& w_{\text{I}} \in \mathbb{R}
 \end{aligned}$$

This linear program is called the **dual program** of (1),  $w_{\text{I}}$  and  $z_j$  are the **dual variables**.

Note that this represents exactly the optimization problem to find the loss-ceiling and an optimal strategy of player II (with  $w_{\text{I}} = v_{\text{II}}$  and  $z_j = y_j$ )!