Algorithmic Game Theory

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Assignment 2

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Exercise 2.1. (2 + 3 Points)

Suppose $(a_{\mathtt{I}}, a_{\mathtt{II}})$ and $(b_{\mathtt{I}}, b_{\mathtt{II}})$ are two mixed Nash equilibria in a 2-player zero-sum game with players I, II and their respective strategy sets $S_{\mathtt{I}}$, $S_{\mathtt{II}}$.

a) Are $(a_{\mathtt{I}}, b_{\mathtt{II}})$ and $(b_{\mathtt{I}}, a_{\mathtt{II}})$ mixed Nash equilibria? Prove your answer.

b) Show that the state $(x_{\mathtt{I}}, x_{\mathtt{II}})$ with

$$x_{\mathtt{I},j} = \frac{1}{2} \cdot (a_{\mathtt{I},j} + b_{\mathtt{I},j}) \qquad \text{for all } j \in \{1, \dots, |S_{\mathtt{I}}|\},$$
$$x_{\mathtt{II},j} = \frac{1}{2} \cdot (a_{\mathtt{II},j} + b_{\mathtt{II},j}) \qquad \text{for all } j \in \{1, \dots, |S_{\mathtt{II}}|\}$$

is a mixed Nash equilibrium.

Exercise 2.2. (2 + 2 + 4 Points)

Consider a 2-player zero-sum game represented by the matrix $A \in \mathbb{R}^{k \times \ell}$. The value of the game is denoted by $v(A)$.

a) For any constant $c > 0$, let $B \in \mathbb{R}^{k \times \ell}$ be a matrix with

$$b_{ij} := c \cdot a_{ij} \qquad \forall (i,j) \in \{1, \dots, k\} \times \{1, \dots, \ell\}.$$

Prove: $v(B) = c \cdot v(A)$ holds for the value $v(B)$ of the 2-player zero-sum game given by B.

b) For any constant $c \in \mathbb{R}$, let $B \in \mathbb{R}^{k \times \ell}$ be a matrix with

$$b_{ij} := c + a_{ij} \qquad \forall (i,j) \in \{1, \dots, k\} \times \{1, \dots, \ell\}.$$

Prove: $v(B) = c + v(A)$ holds for the value $v(B)$ of the 2-player zero-sum game given by B.

c) Let $B \in \mathbb{R}^{k \times \ell}$ be an arbitrary matrix, representing a 2-player zero-sum game with value $v(B)$. Prove or disprove:

$$v(A + B) = v(A) + v(B).$$

Exercise 2.3. (1 + 3 Points)

For an exact potential game Γ with state space Σ, show that

a) there are infinitely many exact potential functions for Γ,

b) for every pair $\Phi_1 \neq \Phi_2$ of exact potential functions for Γ, there is a constant c_{12}, such that

$$\Phi_1(S) - \Phi_2(S) = c_{12} \quad \text{for all states } S \in \Sigma.$$

Exercise 2.4. (4 Points)

Due to increased expenses for electricity, the city council came up with an idea: Instead of having streetlights that burden the city's budget, each house owner is required to install an outdoor lamp to light the street. House owners pay individually for the electricity of their lights and can control if it's switched on or off.

Not every light might be needed for a sufficiently lighted street. However, if there are too many switched off lights in the same area, the city council expects a negative impact. Therefore, house owners have to pay a fine when their light is switched off. The amount depends on the number of neighboring houses who also switched off their light.

The problem is modeled as follows: The neighborhood is given as an undirected graph $G = (V, E)$ where V is the set of houses. Two vertices are neighbors in the graph (i.e. connected with an edge) if and only if the houses are neighboring in the real city.
Each $v \in V$ represents a player in a strategic game with the strategy space $\Sigma_v = \{\texttt{ON}, \texttt{OFF}\}$.
For simplicity, electricity cost is given by a constant $\kappa > 0$. Furthermore, there is a constant $\alpha > 0$ for the exact fine. The cost function for each $v \in V$ in any state S is then given by

$$c_v(S) = \begin{cases} \kappa & \text{, if } S_v = \texttt{ON}, \\ \alpha \cdot \left| \{ \{u, v\} \in E \ : \ S_u = \texttt{OFF} \} \right| & \text{, if } S_v = \texttt{OFF}. \end{cases}$$

Show that this game is an exact potential game.

Assignments and further information concerning the course can be found at
`http://algo.cs.uni-frankfurt.de/lehre/agt/winter2223/agt2223.shtml`

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