# Algorithmic Game Theory

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## Assignment 3

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Due: Nov 15, 2022, **10:00h**

**Exercise 3.1.** (2 + 3 Points)

a) The bimatrix game *battle of sexes* is defined by the following matrix.

| | Zeil | Eintracht |
|---|---|---|
| Zeil | 1<br>2 | 6<br>6 |
| Eintracht | 5<br>5 | 2<br>1 |

Construct an exact potential function for this game and prove its correctness.

b) Construct a $2 \times 2$ bimatrix game with a pure Nash equilibrium and without exact potential function. Prove the correctness of your construction.

**Exercise 3.2.** (2 + 2 + 2 Points)

a) Calculate all values of $\alpha$ for which the function $d_r : \{1, \ldots, n\} \to \mathbb{N}$ with $d_r(x) = x^2$ is an $\alpha$-bounded jump.

b) Let $d : \{1, \ldots, n\} \to \mathbb{N}$ be a positive, monotonically increasing function with *bounded slope*, i.e. for any $x_1, x_2 \in \{1, \ldots, n\}$ and some constant $K$, it holds:

$$|d(x_1) - d(x_2)| \leq K \cdot |x_1 - x_2|.$$

Prove or disprove: $d$ is a $(K + 1)$-bounded jump.

c) Let $d : \{1, \ldots, n\} \to \mathbb{N}$ be a $(K + 1)$-bounded jump, for some constant $K$.

Prove or disprove: For all $x_1, x_2 \in \{1, \ldots, n\}$, it holds:

$$|d(x_1) - d(x_2)| \leq K \cdot |x_1 - x_2|.$$

**Exercise 3.3.** (3 + 3 Points)

Consider the game described in Exercise 2.4 from the previous exercise sheet:

There is a given graph $G = (V, E)$ and given constants $\kappa > 0$ and $\alpha > 0$. Each $v \in V$ represents a player with strategy set $\Sigma_v = \{\texttt{ON}, \texttt{OFF}\}$ whose costs in any state $S$ are given by

$$c_v(S) = \begin{cases} \kappa & \text{, if } S_v = \texttt{ON}, \\ \alpha \cdot \big|\{\ \{u,v\} \in E \ :\ S_u = \texttt{OFF}\}\big| & \text{, if } S_v = \texttt{OFF}. \end{cases}$$

a) Construct an isomorphic congestion game with strategy space $\Sigma_i = \{\texttt{ON}_i, \texttt{OFF}_i\}$ for all players $i \in \mathcal{N}$. Prove the correctness of your construction.

   *Hint*: Define resources $\mathcal{R}$, delays $(d_r(n_r))_{r\in\mathcal{R}}$ and the strategy space $\texttt{ON}_i, \texttt{OFF}_i \subseteq \mathcal{R}$ for each $i \in \mathcal{N}$ appropriately and show that every player has identical costs in corresponding states of both games.

b) Prove that every sequence of best response improvement steps has length $\mathcal{O}\left(|V|^3\right)$.

   What is the best upper bound that you can prove?