Algorithmic Game Theory

Winter Term 2022/2023

Prof. Dr. Martin Hoefer, Dr. Giovanna Varricchio
Conrad Schecker

Institute for Computer Science
Algorithms and Complexity

GOETHE UNIVERSITÄT FRANKFURT AM MAIN

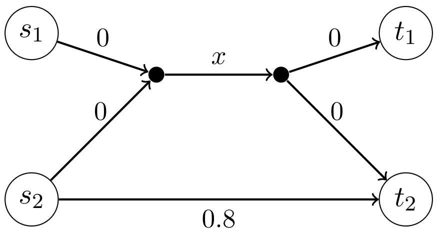
Assignment 7

Issued: Dec 06, 2022
Due: Dec 13, 2022, **10:00h**

This is the last assignment that is relevant for the first part of this course.

Exercise 7.1. (1 + 4 + 2 Points)

Consider the following multi-commodity Wardrop game. The graph and the latency functions are given as depicted below. There are two commodities, (s_1, t_1) with $r_1 = 0.4$ and (s_2, t_2) with $r_2 = 0.6$.



a) Consider the following flow: All players of commodity 1 use the only possible path. The players of commodity 2 split as follows: A total mass of 0.4 uses the direct edge from s_2 to t_2 and the rest uses the upper path.

   Calculate the social cost for the given profile.

b) Calculate a Wardrop equilibrium, the socially optimal flow, and the prices of anarchy and stability.

c) Now assume that the players in every commodity form a coalition and chose their paths in such a way that they minimize the total cost of the population in their respective commodity (this model is also called *atomic-splittable routing game*).

   Calculate the equilibrium flow and the social cost of the equilibrium flow for this situation.

Exercise 7.2. (3 + 3 Points)

a) Construct an equal-sharing game with a pure Nash equilibrium S and a socially optimal state S^* such that

$$\frac{\text{cost}(S)}{\text{cost}(S^*)} = n \ .$$

   Argue in one sentence why S is a pure Nash equilibrium.

b) Prove that equal-sharing games are $(n, 0)$-smooth, i.e. that the price of anarchy for coarse-correlated equilibria is at most n.

Exercise 7.3. (2 + 2 + 3 Points)

We consider the following model for a vertex cover resource buying game:

There is an undirected graph $G = (V, E)$. Each edge $e \in E$ is a player that needs to be covered. The vertices in V are resources. Every $v \in V$ has a cost of $c_v \geq 0$. Every $e \in E$ chooses at least one of its incident vertices. We denote the choice of e by S_e.

We consider the natural **equal sharing** and **arbitrary sharing** variants of this game. In the equal sharing variant, every v that is chosen by at least one edge becomes bought. The cost c_v is shared equally by all players choosing v. In the arbitrary sharing variant, v is bought if the sum of all payments of incident players for a node v reaches c_v – otherwise, v is not bought, and every $e \in E$ with $v \in S_e$ suffers infinite cost.

In both variants, it is easy to see that the set of resources bought in a pure Nash equilibrium (social optimum state) represents a(n optimal) vertex cover.

Let $n = |E|$ denote the number of players. Prove the following statements for pure Nash equilibria:

a) For both variants, there are instances where the Price of Anarchy is at least n.

b) For the equal sharing variant, there is an instance where the Price of Stability is arbitrarily close to the n-th harmonic number $\mathcal{H}_n$.

c) For the arbitrary sharing variant, the Price of Stability is exactly 1.