Algorithmic Game Theory

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Assignment 12

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Due: Feb 07, 2023, **10:00h**

Exercise 12.1. (3 + 2 Points)

Consider the cake-cutting problem on a cake C with valuation functions $v_1, \dots, v_n$ and denote the set of possible allocations by $\mathcal{A}(C)$. The space of Nash optimal allocations is then given by $\arg\max_{\mathcal{A} \in \mathcal{A}(C)} \{NSW(\mathcal{A})\}$.

a) Construct an instance for the cake-cutting problem with the following constraints:

- There is an allocation $\mathcal{A}' \in \mathcal{A}(C)$ that is EF and PO, but $\mathcal{A}' \notin \arg\max_{\mathcal{A} \in \mathcal{A}(C)} \{NSW(\mathcal{A})\}$.
- There is an allocation $\mathcal{A}^* \in \arg\max_{\mathcal{A} \in \mathcal{A}(C)} \{NSW(\mathcal{A})\}$ that is both EF and PO.

Prove the correctness of your construction.

b) For any $i \in \mathcal{N}$, $\lambda_i > 0$, $A_i \subseteq C$, let

$$v_i'(A_i) := \lambda_i \cdot v_i(A_i)$$

denote valuation of agent i for A_i, scaled by a factor of λ_i.

Prove that the space of Nash optimal allocations for valuations $v_1', \dots, v_n'$ is identical to the space of Nash optimal allocations for valuations $v_1, \dots, v_n$.

Exercise 12.2. (3 Points)

Prove that an allocation obtained with the Selfridge-Conway protocol is not necessarily Pareto-optimal. Explain the allocation that is chosen by the protocol in detail!

Exercise 12.3. (2 + 2 Points)

Consider fair division of indivisible goods where all agents have additive, binary valuations, i.e., $v_i(g) \in \{0, 1\}$ for all $i \in \mathcal{N}$, $g \in \mathcal{G}$. Let $\mathcal{A} = (A_1, \dots, A_n)$ be an allocation that maximizes Nash social welfare. You may assume that under all such allocations, $\mathcal{A}$ first maximizes the number of agents having a positive valuation and then the Nash social welfare among these agents.

Prove the following statements:

a) For every $j \in \mathcal{N}$ and $g \in A_j$ with $v_j(g) = 0$, it holds $v_i(g) = 0$ for all $i \in \mathcal{N}$.

b) Let $i, j \in \mathcal{N}$. If i envies j, then it holds $v_i(A_j) = v_i(A_i) + 1$.

Exercise 12.4. (2 + 3 Points)

Consider the Round-Robin procedure for fair division of m indivisible goods.

a) For which number of goods m is the procedure incentive-compatible? Prove your answer.

b) We consider another sequencial algorithm with sequence $s = (s_1, \ldots, s_m)$ that has the following properties:

- For every $k \in \{1, \ldots, \lfloor \frac{m}{n} \rfloor\}$, the partial sequence $(s_{n(k-1)+1}, s_{n(k-1)+2}, \ldots, s_{n \cdot k})$ is an arbitrary permutation of $\mathcal{N}$, i.e., it contains all n agents in an arbitrary order.
- All the components of the remaining partial sequence $(s_{n\lfloor \frac{m}{n} \rfloor+1}, s_{n\lfloor \frac{m}{n} \rfloor+2}, \ldots, s_m)$ are pairwise distinct.

Prove that this algorithm produces an EF1 allocation under additive valuations.