Algorithmic Game Theory

Winter Term 2023/2024

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Assignment 1

Issued: Oct 24, 2023
Due: Oct 30, 2023, **23:55h**

General Information on Submissions

Every submission for this course...

- ... must consist of a **single PDF file**.
- ... must be uploaded **latest on Monday, 23:55h** in the week after the assignment was issued. You should have received a personal upload URL after signing up for the exercises.
- ... can be composed in English or German.
- ... will be graded based on correctness, completeness, comprehensibility and conciseness. In particular, **all answers require an explanation**, unless stated otherwise.

Exercise 1.1. (2 + 4 Points)

	E	F	G	H
A	7 / 4	8 / 5	2 / 9	9 / 4
B	8 / 6	4 / 6	1 / 8	5 / 3
C	3 / 3	7 / 5	5 / 4	3 / 1
D	6 / 5	6 / 5	4 / 7	9 / 8

Consider the 2-player game given by the cost matrix above. Calculate all

a) dominant strategies of the players,

b) pure Nash equilibria.

Exercise 1.2. (3 + 3 + 3 Points)

A strategy $s_i \in S_i$ of player i is called *strictly dominated* by strategy $s_i' \in S_i$, if s_i' is always strictly better then s_i, i.e. for all s_{-i} we have $c_i(s_i', s_{-i}) < c_i(s_i, s_{-i})$.

	W	X	Y	Z
A	3, 1	9, 1	5, 2	6, 2
B	5, 2	4, 5	2, 3	9, 7
C	2, 7	7, 9	9, 5	1, 6
D	2, 7	8, 6	3, 5	4, 8

a) Iteratively eliminate all strictly dominated strategies in the cost matrix given above. Do this until there are no strictly dominated strategies left. Depict your cost matrix after each step.

b) Calculate a mixed Nash equilibrium for the reduced cost matrix.

c) Prove the following statement: In all 2-player normal form games, there is a one-to-one correspondence between mixed Nash equilibria in the original and the reduced game, when applying the reduction procedure described in a).

Exercise 1.3. (4 Points)

The *game of depression* is given by a triple $\left(\mathcal{N}, \mathcal{A}, (c_{ij})_{(i,j)\in\mathcal{N}\times\mathcal{A}}\right)$, where

- $\mathcal{N} = \{1, 2, \dots, n\}$ is the set of *players*, every player has 1 Euro of money,
- $\mathcal{A} = \{1, 2, \dots, m\}$ is the set of *assets* on which players can spend money on,
- $c_{ij} \geq 0$ is the *intrinsic cost value* of asset $j \in \mathcal{A}$ for agent $i \in \mathcal{N}$.

The goal of each player is to divide her one Euro among the assets in a way that minimizes her cost. An *investment strategy* of player i is a tuple $x_i \in [0,1]^m$ with $\sum_{j=1}^m x_{ij} = 1$, where x_{ij} is the amount of money player i puts on asset j. Given the investment strategies $(x_i)_{i\in\mathcal{N}}$ of each player, the cost of player i is given by the function

$$c_i(x) = \sum_{j\in\mathcal{A}} x_{ij} c_{ij} \cdot \prod_{i'\in\mathcal{N}\setminus\{i\}} x_{i'j}^2.$$

Note that the function $c_i(x_i, x_{-i})$ is linear in x_i.

A strategy profile $(x_i)_{i\in\mathcal{N}}$ is called *investment equilibrium* if for every player $i \in \mathcal{N}$, there is no investment strategy x_i' such that $c_i(x_i', x_{-i}) < c_i(x_i, x_{-i})$.

Use Brouwer's fixed point theorem to show that in every depression game there exists an investment equilibrium. It is sufficient to give a short explanation why the prerequisites of Brouwer's fixed point theorem are satisfied.

Exercise 1.4. (3 + 3 Points)

Let $(\mathcal{N}, (S_i)_{i\in\mathcal{N}}, (c_i)_{i\in\mathcal{N}})$ be a game in normal form where $\mathcal{N} = \{1, 2\}$ and $S_1 = \{A, B\}$, that is, there are only two players and the first one has only two strategies. The set of strategies of the second player is $S_2 = \{1, 2, \ldots, k\}$ where $k \in \mathbb{N}$.

Claim 1: In the above game, there always exists a mixed Nash equilibrium x in which player 2 mixes at most two strategies, that is, $|\{j \in S_2 \mid x_{2j} > 0\}| \leq 2$.

a) Describe an algorithm to compute a mixed Nash equilibrium for the game. The algorithm should run in polynomial time in the input size. You can make use of **Claim 1**.

b) Prove **Claim 1**.