Algorithmic Game Theory

Winter Term 2023/2024

Prof. Dr. Martin Hoefer
Marco Schmalhofer, Lisa Wilhelmi

GOETHE UNIVERSITÄT FRANKFURT AM MAIN
Institute for Computer Science
Algorithms and Complexity

Assignment 4

Issued: Nov 14, 2023
Due: Nov 20, 2023, **23:55h**

Exercise 4.1. (2 Points)

A state of a game is called *strong* Nash equilibrium if there is no coalition of players that can improve by simultaneously deviating to another strategy. Formally, a state S of a game is a strong Nash equilibrium if there is no set of players $B \subseteq \mathcal{N}$ such that the players in B have a strategy profile $S'_B = (S'_i)_{i \in B}$ that satisfies $c_i(S'_B, S_{-B}) \leq c_i(S)$ for all $i \in B$ and at least one inequality is strict.

Show that there is a strong Nash equilibrium in every correlated matching game.

Exercise 4.2. (2 + 2 + 2 Points)

Consider the following 2-player bimatrix game.

	E	F	G
A	(2, 2)	(9, 2)	(1, 9)
B	(2, 9)	(2, 2)	(2, 1)
C	(1, 1)	(9, 2)	(2, 2)

The game is played repeatedly. Suppose the players choose the following sequence of strategies:

$$\begin{pmatrix} A \\ E \end{pmatrix}, \begin{pmatrix} B \\ F \end{pmatrix}, \begin{pmatrix} C \\ G \end{pmatrix}, \begin{pmatrix} A \\ E \end{pmatrix}, \begin{pmatrix} B \\ F \end{pmatrix}, \begin{pmatrix} C \\ G \end{pmatrix}, \dots$$

a) Show that this sequence fulfills the no-regret property for both players.

b) Prove or disprove: The average strategies of the two players in the given sequence converges to a mixed Nash equilibrium.

c) Let $\mathcal{V}$ be the probability distribution over states with $\Pr_{s \sim \mathcal{V}}[s] = \frac{1}{3}$ for $s \in \left\{ \binom{A}{E}, \binom{B}{F}, \binom{C}{G} \right\}$ and 0 otherwise.

   Prove or disprove: $\mathcal{V}$ is a coarse-correlated equilibrium.

Exercise 4.3. (3 + 3 Points)

Suppose Γ is a finite normal-form game, in which every player i has a strictly dominant strategy $s_i^{DS} \in S_i$, i.e.,

$$c_i(s_i, s_{-i}) > c_i(s_i^{DS}, s_{-i}) \qquad \text{for all } s_i \in S_i \setminus \{s_i^{DS}\}, s_{-i} \in S_{-i} \; .$$

We call such a game *strictly dominant*. Let s^{DS} be the dominant-strategy equilibrium. Consider the distribution $\mathcal{V}^{DS}$ with $\Pr_{s \sim \mathcal{V}^{DS}}[s] = 1$ if $s = s^{DS}$ and 0 otherwise.

Prove or disprove the following statements:

a) $\mathcal{V}^{DS}$ is a coarse-correlated equilibrium in every strictly dominant game Γ.

b) There are strictly dominant games Γ with coarse-correlated equilibria $\mathcal{V} \neq \mathcal{V}^{DS}$.

Exercise 4.4. (3 + 2 + 2 Points)

Consider a variant of the expert problem with N experts and two events $\{Even, Odd\}$. In every week, each expert predicts if the sum of points on this week's AGT sheet is even or odd. Denote with S_e^t the number of experts predicting *Even* for the t-th sheet.

Before publication of the t-th sheet, the algorithm PointJority selects a prediction based on the experts' history and current guess. Therefore, expert i has a weight $w_i^{t-1} \in [0, 1]$ before sheet t is published, starting with $w_i^0 = 1$ for all experts. Let $W^{t-1} = \sum_{i=1}^{N} w_i^{t-1}$ be the sum of all weights before sheet t is published. The algorithm makes a majority decision based on weights, that is, it predicts *Even* for the t-th sheet if

$$\sum_{i \in S_e^t} w_i^{t-1} \geq W^{t-1}/2 \; ,$$

and *Odd* otherwise.

After publication of sheet t, the algorithm updates the weights of the experts: The weight of every expert i who gave a wrong prediction gets halved, i.e., $w_i^t = w_i^{t-1}/2$.

a) Prove that if PointJority makes a wrong prediction for sheet t, then

$$W^t \;\leq\; \frac{3}{4} \cdot W^{t-1} \; .$$

b) Let f be the overall number of wrong predictions of PointJority after T sheets have been published. Prove that

$$f \;\leq\; \log_{4/3}\left(\frac{W^0}{W^T}\right) .$$

c) Let f_i be the overall number of wrong predictions of expert i. Use the statement in b) to prove that for each $i \in N$,

$$f \;\leq\; \frac{1}{\ln(4/3)} \cdot (f_i + \ln N) \; .$$