Algorithmic Game Theory

Winter Term 2023/2024

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Assignment 6

Issued: Nov 28, 2023
Due: Dec 04, 2023, **23:55h**

- Part 1 will end with lecture 14 on Dec 7.
- This is the last assignment of part I and be discussed in the exercise session on Dec 8.

Exercise 6.1. (1 + 3 + 3 + 2 + 2 Points)

In a *connection game with high edge costs* there is a set $\mathcal{N}$ of $n > 1$ players and edge costs $\alpha > n(n-1)$. Each player is a node. Player i chooses a subset $S_i \subseteq \mathcal{N} \setminus \{i\}$ as strategy. If $j \in S_i$, then i buys an *undirected edge* $\{i, j\}$ to j. If an edge is bought by one player, it can be used by all players.

The *network in state* S is $G(S) = (\mathcal{N}, E(S))$, a multigraph of all player nodes and all purchased edges $E(S) = \{\{i, j\} \mid i \in \mathcal{N}, j \in S_i\}$.

For his costs, player i considers the costs $\alpha \cdot |S_i|$ of the edges he buys and the length $dist_{G(S)}(i, j)$ of a shortest path in $G(S)$, for every other player $j \in \mathcal{N}$. If there is no path between i and j, then $dist_{G(S)}(i, j) = \infty$. The costs of i in state S are

$$c_i(S) = \alpha \cdot |S_i| + \sum_{j \in \mathcal{N}} dist_{G(S)}(i, j) \ .$$

The social costs are $cost(S) = \sum_{i \in \mathcal{N}} c_i(S)$.

Show the following statements:

a) Consider a connection game with high edge costs and 5 agents. Derive a pure Nash equilibrium that is not a social optimum.

b) For every pure Nash equilibrium S of a connection game with high edge costs, the network $G(S)$ is a tree.

c) For every social optimum S^* of a connection game with high edge costs, the network $G(S)$ is a star graph (i.e. there is a node i' and $E(S^*) = \{\{i', j\} \mid j \in \mathcal{N} \setminus \{i'\}\}$).

d) The price of anarchy for pure Nash equilibria in connection games with high edge costs is at most 2.

e) There are no constants $\lambda > 0$ and $\mu < 1$, so that every connection game with high edge costs is (λ, μ)-smooth.

Exercise 6.2. (3 + 3 Points)

a) Construct an equal-sharing game with a pure Nash equilibrium S and a socially optimal state S^* such that

$$\frac{\text{cost}(S)}{\text{cost}(S^*)} = n \ .$$

Argue in one sentence why S is a pure Nash equilibrium.

b) Prove that equal-sharing games are $(n, 0)$-smooth, i.e. that the price of anarchy for coarse-correlated equilibria is at most n.

Exercise 6.3. (3 + 4 Points)

a) Construct a congestion game with affine linear delay functions and price of stability arbitrarily close to $\frac{4}{3}$. Argue why your example has the desired properties.

b) Prove that the price of stability in congestion games with affine linear delay functions is at most 2.

Remark: Assume an affine linear function f to have the following form $f(x) = a \cdot x + b$ for $a, b \geq 0$.

Assignments and further information concerning the course can be found at
`https://algo.cs.uni-frankfurt.de/lehre/agt/winter2324/agt2324.shtml`

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