Algorithmic Game Theory

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Prof. Dr. Martin Hoefer
Marco Schmalhofer, Lisa Wilhelmi

GOETHE UNIVERSITÄT FRANKFURT AM MAIN
Institute for Computer Science
Algorithms and Complexity

Assignment 9

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Exercise 9.1. (4 Points)

Consider the following instance of a multi-item market with unit-demand bidders. The set of bidders is $I = \{1, 2, 3, 4\}$ and the set of items is $J = \{a, b, c\}$. The valuation v_{ij} bidder i has for item j is given by the following table:

	a	b	c
1	6	8	5
2	0	7	6
3	1	9	4
4	5	4	8

Execute the Online VCG Mechanism with sample size $r = 1$ on this instance assuming bidders arrive in the order 1,2,3,4. For every round $t > r$, depict the matchings M_t, M and M'_t and determine the item assigned to bidder t (if any) as well as the payment p_t.

Exercise 9.2. (4 Points)

Prove that the Random Threshold Mechanism is ex-post universally truthful.

Exercise 9.3. (4 Points)

Consider a forest auction with n bidders in a graph G. Assume all bidders have pairwise distinct values. Denote the size of each spanning tree in G by k and use the Random Threshold Mechanism. Let S denote the output of the Random Threshold Mechanism and $m_i(S)$ the number of bidders in S with values at least $v_i/2$, where v_i is the i-th highest value of the bidders in the optimal spanning tree. Prove that

$$\mathbb{E}[m_1(S)] \geq \frac{1}{8(\lceil \log k \rceil + 1)} .$$

Exercise 9.4. (4 + 4 + 4 Points)

Let $G = (K, E)$ be a graph where every edge $e \in E$ has a value $v_e \geq 0$. A matching is a set $M \subseteq E$ of edges such that no vertex $x \in K$ is incident to more than one edge from M, i.e., $|\{e \in M : x \in e| \leq 1$. Let the set of outcomes A be the set of all matchings in G, and denote by M^* a matching of maximum value.

a) The (offline) *Greedy Algorithm* starts with $M_g = \emptyset$ and iterates over the edges in non-increasing order of value. An edge is added to M_g if M_g is still a matching afterwards. Show that for the resulting matching M_g it holds

$$v(M_g) \geq \frac{1}{2} \cdot v(M^*) \ .$$

In the Random-Order model, initially all values are unknown. The edges reveal their value one by one in uniform random order. An online algorithm must immediately and irrevocably accept or reject an edge before getting to know the value(s) of the next edge(s). The goal is to accept a matching as good as possible.

The Random Threshold Mechanism with $k = \max_{M \in A} |M|$ can be directly applied to this setting. We aim to show that it has the same guarantee as for forest auctions.

b) For the analysis, number the edges in M_g from 1 to $|M_g|$ according to the ordering they are selected by the greedy algorithm in a). As in the lecture, define q and $m_i(S)$ regarding M_g (instead of S^*).

   Show that for the (random) matching S computed by the Random Threshold Mechanism, it holds for all $1 \leq i \leq q$,

$$\mathbb{E}[m_i(S)] \geq \frac{1}{16(\lceil \log k \rceil + 1)} \cdot i \ .$$

   It is sufficient to describe how the original proof from the lecture for $i > 1$ needs to be adapted using the result from a).

c) Use the result from a) and b) to prove that the Random Threshold Mechanism is $O(\log k)$-competitive in this setting.