Algorithmic Game Theory

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Assignment 10

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Due: Jan 29, 2024, **23:55h**

Exercise 10.1. (2 + 4 + 3 Points)

For the PROPHET setting, let q_i be the probability that v_i is optimal, where $i = 1, ..., n$. Furthermore, suppose that τ_i is such that $\Pr[v_i \geq \tau_i] = q_i$, i.e., the q_i^{th} percentile for v_i. For simplicity, assume that such a threshold τ_i always exists. Define

$$\tilde{v}_i(q_i) := \mathbb{E}[v_i \mid v_i \geq \tau_i]$$

as the expected value of v_i given that it lies in the top q_i^{th} percentile. We consider the following mechanism to assign the item: *When bidder i shows up, if the item has not been allocated to $1, ..., i-1$, reject her with probability 1/2 outright, else allocate the item if $v_i \geq \tau_i$ with payments $p_i = \tau_i$.*

a) Show that the described mechanism is incentive compatible.

b) Show that the mechanism achieves a value of at least $\frac{1}{4} \cdot \mathbb{E}[v_{\max}]$.

   Hint: Show that the algorithm gets value $\frac{1}{4} \cdot \sum_i \tilde{v}_i(q_i) \cdot q_i \geq \frac{1}{4} \cdot \mathbb{E}[v_{max}]$. Use Markov's inequality to bound the probability that at least one bidder has been allocated to the item so far.

c) Suppose we have k identical items and can allocate up to k bidders to an item. Thus, the sequence ends with allocation of the k-th item or after the arrival of the n-th bidder. The quantity of interest is given by $\mathbb{E}[\text{sum of values of}\, k\, \text{highest bids}]$. Show that the algorithm achieves at least $\frac{1}{4}$ of it.

   Hint: Redefine q_i as the probability that v_i is among the top k bidders.

Exercise 10.2. (2 + 2 Points)

a) Show that the plurality rule does not always fulfill the Condorcet-winner criterion.

b) Show that a social choice function is incentive compatible (IC) if and only if it is monotone.

Exercise 10.3. (4 + 2 Points)

Consider the following social choice function with n voters and $|A| > 1$. Derive a candidate who was ranked in first position the least and eliminate this candidate. Break ties in alphabetical order. If one of the remaining candidates was ranked in first place by more than $n/2$ voters, elect the candidate. Otherwise, proceed by eliminating the next candidate.

a) Which of the following properties is satisfied by this social choice function? Prove your answer.

- Monotonicity
- Condorcet-Winner Criterion

b) Consider the corresponding social welfare function F which defines a preference $\succ$ over all candidates. The preference $\succ$ is determined by iteratively applying the social choice function f. In the first iteration, $f(\succ_1, \ldots, \succ_n) = a$ derives the winning candidate. Place $a \in A$ in the first position of $\succ$. Remove a from A and apply f on the remaining preferences. Remove the winner from A and place the candidate in the second highest position in $\succ$. Repeat this process until only one candidate is left. This candidate is assigned the last position in $\succ$.

   Prove or disprove: The social welfare function satisfies the Independence of Irrelevant Alternatives property.

Exercise 10.4. (2 + 2 Points)

We consider a social choice function that iteratively eliminates candidates to select a winner. In each step, an arbitrary pair of (remaining) candidates is compared using the majority rule and the candidate that is not the winner is eliminated. The process stops when there is no pair of candidates remaining such that one candidate can be eliminated. If only a single candidate remains, she is then selected as the winning candidate. Otherwise, there is not a winner. Note that this social choice function (i.e. the order of comparisions) can be represented by a tournament tree.

a) Prove: There is always a winner (for every possible tournament tree) if and only if n is odd.

b) Consider the following instance with candidate set $A = \{a, b, c, d\}$:

$$a \succ_1 b \succ_1 c \succ_1 d,$$
$$c \succ_2 d \succ_2 b \succ_2 a.$$

   Construct the preference order $\succ_3$ of a third voter that fulfills both of the following requirements:

   - For each candidate $x \in A$, there is a tournament tree where x wins.
   - The Borda Count of each pair of candidates differs by at most 1.

   Prove the correctness of your construction.