Approximation Algorithms

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Assignment 5

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Exercises marked with * are bonus - they count for your score but not for the sum of points.

Exercise 5.1 k-Center *Taking Center Stage* (3 + 3 + 3* points)

a) Let $c = 1 + \varepsilon$ for some positive $\varepsilon < 0.001$. The set $K = \{A, B, C, D, E, F\}$ of six data points in the Euclidean plane defines an input for the 2-Center problem, where

$$A = (0,0), \quad B = (c,0), \quad C = (2c,0), \quad D = (0,2), \quad E = (c,2), \quad F = (2c,2).$$

Using the 2-flip neighborhood $\mathcal{N}_2(M) := \{M' \mid |(M' \setminus M) \cup (M \setminus M')| \leq 2\}$, determine a locally optimal solution $M \subseteq K$ for 2-Center that is not globally optimal.
Hint: The parameter ε may be neglected if its size is not decisive.

b) Show that the running time of the *Strict Local Search* algorithm on k-Center with the 2-flip neighborhood is in poly(n).
Hint: How many possible improvements exist at most until a locally optimal solution is found?

c)* For a set K of n data points and a metric $d : K \times K \to \mathbb{R}_{\geq 0}$, let r^* be the maximum distance of any point to its closest center in the optimal solution C^* for k-Center on K. Consider the following algorithm for a given parameter $\ell \geq 0$ which computes a set C of centers:

1. Initially, label all $x \in K$ as unmarked and set $C = \emptyset$.
2. While there are unmarked data points in K, Do:
   Add some arbitrary unmarked $x \in K$ to C and mark all points $x' \in K$ with distance $d(x, x') \leq 2\ell$.

Show: If $\ell \geq r^*$, then $|C| \leq k$.

Exercise 5.2 *Dominating Sets in the Hypercube* (2 + 3 points)

A hypercube is a d-dimensional analogue of a square ($d = 2$) and a cube ($d = 3$). It has 2^d corners. For this exercise, consider the d-dimensional hypercube modeled as a graph $W_d(V, E)$. More specifically, each corner of the hypercube is modeled as a vertex in W_d and is uniquely described by a binary vector of length d. Thus, $V = \{0,1\}^d$. Two vertices $x, y \in \{0,1\}^d$ are connected by an edge in E if and only if x and y differ in exactly one entry. Recall that a set $D \subseteq V$ is *dominating* if, for every $v \in V$, D contains v or at least one vertex adjacent to v.

a) Show that any dominating set D in $W_d(V, E)$ consists of at least $\frac{2^d}{d+1}$ vertices.

b) Construct an optimal dominating set $D^* \subseteq V$ in the 4-dimensional hypercube W_4, i.e., $|D^*|$ is as small as possible. Show that the constructed set is optimal and correct indeed (for the latter, a conclusive sketch of W_4 indicating D^* is sufficient).

Exercise 5.3 *Neighborhoods and Approximation Ratios* (2 + 3 + 4 points)

In the Bipartite Weighted MaxMatching problem, a bipartite graph $G = (L \cup R, E)$ is given together with a weight function $w : E \to \mathbb{R}$ (recall that $L \cap R = \emptyset$ and $E \subseteq L \times R$). The goal is to compute a matching $M^* \subseteq E$ of maximum weight, i.e., $w(M^*) = \sum_{e \in M^*} w(e)$ is maximal. Given a matching M, its k-flip neighborhood is defined by $\mathcal{N}_k(M) := \{M' \mid |(M' \setminus M) \cup (M \setminus M')| \leq k\}$.

Prove the following statements on the approximation ratio of the *Strict Local Search* algorithm with k-flip neighborhood on the Bipartite Weighted MaxMatching problem.

a) For $k = 2$, the approximation ratio cannot be bounded by any constant.

b) For $k = 3$, the approximation ratio is *at least* 2.

c) For $k = 3$, the approximation ratio is *at most* 2.