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Assignment 6

Issued: 30.11.2021
Due: 07.12.2021, **10:15h**

The scope of this assignment is halved due to the absence of the lecture on Nov 23.

Exercise 6.1 *Crude Rounding for* FacilityLocation (2 + 2 + 3 points)

Given a set C of clients and a set F of possible locations for service facilities, the *integer* linear program (ILP) for metric FacilityLocation is as follows:

$$\begin{array}{lll} \text{Minimize} & \sum_{i\in F} f_i\, y_i + \sum_{i\in F}\sum_{j\in C} d_{ij}\, x_{ij} & \\ \text{subject to} & \sum_{i\in F} x_{ij} \geq 1, & \forall j \in C, \\ & x_{ij} \leq y_i, & \forall i \in F, \forall j \in C, \\ & y_i \in \{0,1\}, & \forall i \in F, \\ & x_{ij} \in \{0,1\}, & \forall i \in F, \forall j \in C. \end{array}$$

Here, the variables $x_{ij} \in \{0,1\}$ and $y_i \in \{0,1\}$ for all $i \in F$ and $j \in C$ are introduced, where $y_i = 1$ if and only if the facility at location i is opened and $x_{ij} = 1$ if and only if client j is supplied by the facility at location i. Moreover, $f_i \geq 0$ is the opening cost of $i \in F$ and $d_{ij} := d(i,j)$ denotes the connection cost of client j to location i.

a) Consider two *partial* linear relaxations of the above ILP, denoted by LP_y and LP_x, where either all y_i are relaxed to $y_i \in [0,1]$ or all x_{ij} are relaxed to $x_{ij} \in [0,1]$. Show why LP_y and LP_x do not produce better solutions than ILP.

Let LP be the *total* linear relaxation of the above ILP where all variables are relaxed, i.e., $y_i \in [0,1]$ and $x_{ij} \in [0,1]$ for all $i \in F$ and $j \in C$. Moreover, let y_i^* for $i \in F$ be an optimal assignment of the variables y_i by LP. Consider the algorithm *CrudeRound* which opens facility at location i with probability y_i^*. In the case that no facility is opened at all, the algorithm opens one of the facilities in F uniformly at random.

b) Construct an input for metric FacilityLocation such that the corresponding relaxed LP has at least one solution where $y_i^* < 1$ for all $i \in F$.
Hint: An instance with $|F| = 2$ and $|C| = 1$ is sufficient.

c) Show that the approximation ratio of *CrudeRound* cannot be upper bounded by a value polynomial in the input size.

Exercise 6.2 *Infinite* FACILITYLOCATION (3 points)

An instance of metric FACILITYLOCATION is given by the discrete infinite grid $\mathbb{Z} \times \mathbb{Z}$. Every grid point corresponds to a client $j \in C$ and is also a possible location for a service facility $i \in F$, i.e., $F = C = \{(a, b) \mid a, b \in \mathbb{Z}\}$. For every facility i, opening costs are given by $f_i = 2$. The underlying metric is the Manhattan distance, i.e., the distance of two grid points (a_1, b_1) and (a_2, b_2) is defined by $d((a_1, b_1), (a_2, b_2)) = |a_2 - a_1| + |b_2 - b_1|$.

Determine an optimal set of facilities $X^* \subseteq F$ that minimizes the *average* cost per grid point. Specify both the best possible frequency of facilities and their positions.