Approximation Algorithms

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Assignment 9

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Exercise 9.1 *Dual LP* (2 + 2 + 3 points)

a) Construct the dual to the following (primal) LP:

$$\begin{array}{lrcl} \text{Minimize} & 2x_2 + 5x_3 & & \\ \text{subject to} & x_1 + x_2 & \geq & 2 \\ & 2x_1 + x_2 + 6x_3 & \leq & 6 \\ & x_1 - x_2 + 3x_3 & = & 4 \\ & x_1, x_2, x_3 & \geq & 0 \end{array}$$

b) An undirected graph $G = (V, E)$ with edge weights $w_e \geq 0$, for all $e \in E$, is given. The goal of the Edge Cover problem is to determine a set of edges $S \subseteq E$ with minimal sum of weights such that S covers every vertex, i.e., there exists an edge $\{v, u\} \in S$, for all $v \in V$.

Derive an LP-relaxation for the Edge Cover problem. The upper bound of 1 for every variable can be omitted.

c) State the dual to the LP derived in b). For which graph problem is the dual an LP-relaxation if the graph is connected and all edge weights are uniform, i.e., $w_e = 1$, for all $e \in E$?

Exercise 9.2 *Approximate Complementary Slackness* (2 points)

The primal LP of a minimization problem and its dual are defined as:

$$\begin{array}{lrcl} \text{Minimize} & \mathbf{c}^T\mathbf{x} & & \\ \text{subject to} & \mathbf{A}\mathbf{x} & \geq & \mathbf{b} \\ & \mathbf{x} & \geq & \mathbf{0} \end{array} \qquad \begin{array}{lrcl} \text{Maximize} & \mathbf{y}^T\mathbf{b} & & \\ \text{subject to} & \mathbf{y}^T\mathbf{A} & \leq & \mathbf{c}^T \\ & \mathbf{y} & \geq & \mathbf{0} \end{array}$$

where $\mathbf{A} \in \mathbb{Q}^{m\times n}, \mathbf{c} \in \mathbb{Q}^n$ and $\mathbf{b} \in \mathbb{Q}^m$. Furthermore, let $\mathbf{x} \in \mathbb{Q}^n$ and $\mathbf{y} \in \mathbb{Q}^m$ denote feasible solutions for the respective problems and assume that there exist positive numbers $\alpha \geq 1, \beta \leq 1$ so that the two following approximate complementary slackness conditions hold:

$$\begin{array}{rcl} x_j > 0 & \Rightarrow & \mathbf{y}^T\mathbf{a}^j \geq \beta \cdot c_j \,, \\ y_i > 0 & \Rightarrow & \mathbf{a}_i \cdot \mathbf{x} \leq \alpha \cdot b_i \,. \end{array}$$

Show that $\mathbf{x}$ is an α/β-approximation for the primal LP.

Exercise 9.3 *Local Ratio* (1 + 2 + 1 + 3 + 4 points)

Local ratio is a technique for designing approximation algorithms. It is closely related to the primal-dual method in linear programming but it uses duality only in an implicit way. The goal of this task is to prove the underlying theorem for this technique and to apply it on the Weighted Set Cover problem for a collection $\mathcal{S}$ of n subsets of the ground set $E = [m]$, where each subset $S_j \in \mathcal{S}$ has weight/cost $w_j \geq 0$.

First, consider the following local ratio algorithm for Weighted Set Cover: Let $C \subseteq [n]$ be a collection of subset indices, where $C = \emptyset$ initially. In each iteration k, the algorithm arbitrarily picks one element e_i that is not yet covered by C. Let τ be the minimum weight of any set containing e_i. For each set S_j containing e_i, the algorithm updates its weight $w_j \leftarrow w_j - \tau$. Consequently, there is now at least one such set that has weight zero. The indices of all these sets are then added to C. The algorithm stops as soon as no uncovered elements remain.

Let τ_k be the value of τ in the kth iteration of the algorithm and let $f > 0$ be the maximum number of sets any single element is contained in.

a) Show that the cost of the solution returned by the algorithm is at most $f \sum_k \tau_k$.

b) Show that the cost of the optimal solution must be at least $\sum_k \tau_k$.

c) Conclude that the algorithm has approximation ratio at most f.

The local ratio technique depends upon the *Local Ratio Theorem*. For a general minimization problem M with a non-negative cost function c as objective function, it can be formulated as follows:

(Local Ratio Theorem) Assume that the cost function of M can be decomposed into $c = c_1 + c_2$, where c_1 and c_2 are also non-negative cost functions. If a feasible solution $\boldsymbol{x}$ is an α-approximation with respect to both c_1 and c_2, then $\boldsymbol{x}$ is also an α-approximation with respect to c.

d) Prove the Local Ratio Theorem.

e) Explain how the algorithm for Weighted Set Cover given above can be analyzed in terms of the Local Ratio Theorem in order to prove that it is an f-approximation algorithm.

The assignments and further information on the course are provided on our website:
https://algo.cs.uni-frankfurt.de/lehre/apx/winter2122/apx2122.shtml

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