Approximation Algorithms
Winter term 2021/22

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Assignment 10

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Exercise 10.1 *Primal-Dual Interpretation of Dijkstra* (3 points)

Consider the primal-dual algorithm for the SHORTEST PATH problem discussed in the lecture. Show that in each step of this algorithm, it selects the same edge as *Dijkstra's* algorithm.

Exercise 10.2 MULTICUT *in Trees* (2 + 4 + 2 + 2 points)

Consider the MULTICUT problem in trees: a tree $T = (V, E)$, k pairs of vertices $(s_i, t_i) \in V \times V$, and costs $c_e \geq 0$, for each edge $e \in E$, are given. The goal is to find a set of edges $F \subseteq E$ with minimum sum of costs such that for all i, s_i and t_i are in different connected components of $G' = (V, E - F)$.

Let P_i be the set of edges in the unique path in T between s_i and t_i. Then, MULTICUT can be formulated as the following ILP:

$$\begin{aligned} \text{Minimize} \quad & \sum_{e \in E} c_e x_e, \\ \text{subject to} \quad & \sum_{e \in P_i} x_e \geq 1 \qquad \forall\, 1 \leq i \leq k, \\ & x_e \in \{0, 1\} \quad \forall\, e \in E. \end{aligned}$$

Suppose that T is rooted at an arbitrary vertex $r \in V$. For any $v \in V$, let $\mathsf{depth}(v)$ be the number of edges in the path from v to r. Moreover, let $\mathsf{lca}(i)$ be the vertex v on the path from s_i to t_i whose depth is minimum. Suppose that the primal-dual method is used to solve this problem, where the dual variable y_i that is increased in each iteration corresponds to the violated constraint that maximizes $\mathsf{depth}(\mathsf{lca}(i))$ *(initialization phase)*. Let F be the set of edges that results from this procedure. Subsequently, each $e \in F$ is considered in *reverse* order of when it was added to F and is removed from F if $F \setminus \{e\}$ is feasible *(pruning phase)*.

Prove that this yields a 2-approximation algorithm for MULTICUT in trees by performing the following steps.

a) State the LP-relaxation (P) of the above ILP as well as the dual of (P).

b) Let $\bar{\boldsymbol{x}}$ with $\bar{x}_e = 1$ if $e \in F$ and $\bar{x}_e = 0$ if $e \notin F$ be the integer solution associated with F. Show that if $\bar{y}_i > 0$, then $|P_i \cap F| \leq 2$.

c) Conclude that the approximation ratio is 2.

d) The pruning phase is essential in order to achieve the result in c). Suppose that the above algorithm terminates right after the initialization phase and show that the approximation ratio is at least $\frac{k+1}{2}$ in this case.

Exercise 10.3 SET COVER *and* FACILITY LOCATION (2 + 2 + 3 points)

Consider the FACILITY LOCATION problem from the lecture for a set C of clients, a set F of locations, and opening costs $f_j \geq 0$, for each $j \in F$. In the *non-metric* version of the problem (henceforth denoted by FL), the distances $d(i,j) \geq 0$ between each pair $(i,j) \in C \times F$ do not necessarily form a metric and thus represent general connection costs $d_{ij} := d(i,j)$. Also, recall the Weighted SET COVER (SC) problem for a collection $\mathcal{S}$ of n subsets of the ground set $E = [m]$, where each subset $S_j \in \mathcal{S}$ has weight/cost $w_j \geq 0$.

a) Show that SC is a special case of FL.

The statement in a) suggests that the Greedy algorithm for SC (Algorithm 21 in lecture notes) might be applied to FL, which would yield an approximation ratio of at most $H_{|C|}$ for FL.

b) Define a transformation that converts a given instance of FL to an instance of SC.
Comment: The transformation does not have to be implementable in polynomial time.

c) Given an instance of FL, explain how the Greedy algorithm can be implemented to compute an $H_{|C|}$-approximate solution to FL in polynomial time.

The assignments and further information on the course are provided on our website:
https://algo.cs.uni-frankfurt.de/lehre/apx/winter2122/apx2122.shtml

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