Approximation Algorithms

Winter term 2021/22

Prof. Dr. Martin Hoefer
Tim Koglin, Lisa Wilhelmi

GOETHE UNIVERSITÄT FRANKFURT AM MAIN
Institute of Computer Science
Algorithms and Complexity

Assignment 11

Issued: 01.02.2022
Due: 08.02.2022, **10:15h**

- Due to maintenance work by studiumdigitale, Moodle will be offline on Feb 7th from 08:00h on. For this reason, you may hand in your solution via e-mail to Lucas Hammer (lucas.hammer@stud.uni-frankfurt.de) if you cannot manage to upload it in time otherwise.
- Exercises marked with * are bonus - they count for your score but not for the sum of points.

Exercise 11.1 *Parametric Pruning* (2 + 1 + 3 + 3* points)

Consider the *metric* k-Center problem in terms of a complete undirected graph $G = (V, E)$ with edge costs $d_e \geq 0$, $e \in E$, satisfying the triangle inequality, and a positive integer $k \leq |V|$. Define $m := |E|$. Beyond that, a *dominating set* in an undirected graph $H = (U, F)$ is a subset $D \subseteq U$ such that every vertex in $U \setminus D$ is adjacent to a vertex in D. Let $\text{dom}(H)$ denote the size of a dominating set in H with minimum cardinality.

A restatement of metric k-Center shows how parametric pruning applies to it: Sort the edges of G in nondecreasing order of cost, i.e., $d_{e_1} \leq d_{e_2} \leq ... \leq d_{e_m}$ w.l.o.g., and let $G_i = (V, E_i)$, where $E_i = \{e_1, e_2, ..., e_i\}$. The metric k-Center problem is equivalent to finding the smallest index i such that G_i has a dominating set of size at most k. In particular, G_i contains k star graphs spanning all vertices (possibly plus additional edges). If i^* is the smallest such index, then $d_{e_{i^*}}$ is the cost of an optimal solution to metric k-Center, which will be denoted by OPT hereafter.

a) For a graph H, define the *square* $H^2 = (U, F')$, where $\{u, v\} \in F'$ if there is a path of length at most 2 between u and v in H (and $u \neq v$). Let I be an independent set in H^2.

   Show that $|I| \leq \text{dom}(H)$.

Now consider the following algorithm for k-Center, which uses the fact that *maximal* independent sets can be computed in polynomial time.

Algorithm 1: Parametric Pruning for k-Center

Input: Family of graphs G_1, ..., G_m
1. Construct G_1^2, G_2^2, ..., G_m^2
2. Compute a maximal independent set M_i in each graph G_i^2
3. Find the smallest index j such that $|M_j| \leq k$
4. **return** *Independent set* M_j

b) Show that $d_{e_j} \leq \text{OPT}$ for j as defined in the algorithm.

c) Show that the algorithm achieves an approximation ratio of 2 for k-Center.

d)* Give a tight example for the previous algorithm and prove its approximation ratio.

Exercise 11.2 *Variants of k*-PARTITION (3 + 3 points)

In the k-PATH PARTITION problem, a complete, undirected graph $G = (V, E)$ is given with edge costs $c_e \geq 0$ that satisfy the triangle inequality, and a constant $k \in \mathbb{N}_{>0}$ such that $|V| = 0 \pmod{k}$. The goal is to find a collection of paths, consisting of exactly k vertices each, with minimal sum of costs and such that each vertex is on exactly one path. In the k-TREE PARTITION problem, the same input as in the k-PATH PARTITION problem is given except that the graph is not necessarily complete and the edge costs do not necessarily obey the triangle inequality. The goal is to find a collection of trees with minimal sum of costs such that each tree has $0 \pmod{k}$ many vertices, and each vertex is in exactly one tree.

a) An instance for k-PATH PARTITION and an α-approximation algorithm for k-TREE PARTITION are given. Derive a 2α-approximation for k-PATH PARTITION by applying the algorithm on the instance.

b) Consider the following ILP for the k-TREE PARTITION problem where $\delta(S)$ is the set of edges crossing the cut $(S, V \setminus S)$:

$$\begin{aligned} \text{Minimize} \quad & \sum_{e \in E} x_e c_e \\ \text{subject to} \quad & \sum_{e \in \delta(S)} x_e \geq 1 \qquad \text{if } c \cdot k < |S| < (c+1) \cdot k \text{ with } c \in \mathbb{N}_{\geq 0} \\ & x_e \in \{0, 1\} \qquad \text{for all } e \in E \end{aligned}$$

Show that there exists a corresponding primal-dual algorithm which yields a 2-approximation.

Exercise 11.3 *Approximating* STEINER FOREST (3 + 2 + 2 + 1 points)

Consider the primal-dual algorithm for STEINER FOREST discussed in the lecture.

a) Show that there exists no constant $\alpha < 2$ such that this algorithm is α-approximative.

In the *metric* STEINER FOREST problem, all edge costs satisfy the triangle inequality. The goal in the remainder of this exercise is to show that any α-approximation algorithm for metric STEINER FOREST also yields an α-approximation to STEINER FOREST. This implies that for an instance of STEINER FOREST it can be assumed w.l.o.g. that the triangle inequality is satisfied. Proceed in the following steps.

b) Show that any instance G of STEINER FOREST can be transformed into an instance G' of metric STEINER FOREST in polynomial time.

c) Show that a solution for G' can be transformed into a solution for G in polynomial time.

d) Show that an α-approximation algorithm for metric STEINER FOREST also yields an α-approximation for STEINER FOREST.