Optimization and Uncertainty

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Assignment 7

Issued: 13.06.2023
Due: 20.06.2023, **10:00h**

Exercise 7.1 *Signaling Schemes* (2 + 2 = 4 points)

Consider the following instance of the Persuade problem between $\mathcal{S}$ and $\mathcal{R}$ as defined in the lecture. There are three boxes and three possible prize-pair vectors, namely $\theta_1 = ((2,6),(3,5),(1,5)), \theta_2 = ((4,4),(5,4),(4,5))$ and $\theta_3 = ((7,1),(6,2),(5,3))$. Each prize-pair vector has a probability of 1/3 to be drawn by nature.

a) State the linear program which derives the optimal signaling scheme φ^* for the given instance. Use the approach described in the proof of Theorem 24.

b) Derive the optimal signaling scheme φ^* from the LP and state the expected reward both for $\mathcal{S}$ and $\mathcal{R}$.

   Hint: The calculation of φ^* does not need to be specified.

Exercise 7.2 *Direct and persuasive schemes* (3 points)

Show Proposition 3 from the lecture notes: *For every signaling scheme φ there is a direct and persuasive scheme φ' such that $\mathcal{S}$ obtains the same expected value in φ and φ'.*

Exercise 7.3 *Random-Order Signaling* (2 + 1 + 2 + 2 + 3 + 3 = 13 points)

In Random-Order Signaling, n prize-pairs are revealed to both the sender $\mathcal{S}$ and the receiver $\mathcal{R}$ at the beginning. Afterwards, the prize-pairs are packed into n boxes. The boxes are closed (which makes them look identical), permuted in uniform random order, and then labeled from 1 to n. Afterwards $\mathcal{S}$ looks into all boxes and then sends a signal to $\mathcal{R}$, who has to choose exactly one of the boxes. If box i with prize-pair (r_i, s_i) is chosen, then $\mathcal{S}$ gets reward s_i and $\mathcal{R}$ gets reward r_i.

a) Show that there is an optimal scheme φ^* for $\mathcal{S}$ that is symmetric.

b) A box i is Pareto-dominated by box j if the contained prize-pair is strictly better for both $\mathcal{S}$ and $\mathcal{R}$, i.e., it holds $r_j > r_i$ and $s_j > s_i$.

   Prove that a box that is Pareto-dominated is never recommended by $\mathcal{S}$ in an optimal, direct scheme.

c) Consider the geometric representation of prize-pairs in the Euclidean plane, as depicted in the example below. Let S be the convex hull of all prize-pairs, and let $\text{int}(S)$ denote the interior of S, i.e., the open convex hull of all prize-pairs.

   Extend the given visualization of the example instance:

   - Mark the area of possible locations of prize-pairs that (are corresponding to boxes that) are Pareto-dominated.
   - Visualize which parts of $\text{int}(S)$ do **not** intersect with that area.

   Explain why boxes with prize-pair in $\text{int}(S)$ that are **not** Pareto-dominated are never recommended by $\mathcal{S}$ in an optimal, direct scheme.

d) Prove: Any direct scheme for Random-Order Signaling is persuasive if and only if it yields an expected utility of at least $r_E := \frac{1}{n} \cdot \sum_{i=1}^{n} r_i$ for $\mathcal{R}$.

e) Describe an algorithm to compute an optimal scheme φ^* in polynomial time. Give an informal argument why your algorithm yields an optimal policy indeed.

   Hint: Use the previous tasks. Which boxes could $\mathcal{S}$ possibly recommend? What is the optimal weighting of the signals for $\mathcal{S}$ such that the scheme is persuasive?

f) Assume $\mathcal{S}$ could choose the best box i for her by herself (instead of sending a signal to $\mathcal{R}$). Show that the ratio of the expected rewards of i and φ^* for $\mathcal{S}$ can be up to n, and this is the worst case.

i	1	2	3	4	5	6	7	8
r_i	1	3	6	7	12	13	14	16
s_i	8	1	10	9	8	5	6	2

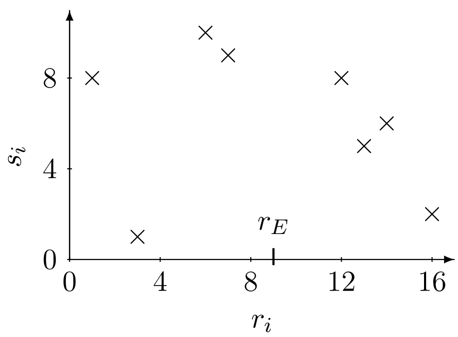

Fig. 1: Example instance with $n = 8$ prize-pairs. Right: Visualization in the Euclidean plane. Each prize-pair is marked by a cross.

The assignments and further information on the course are provided on our website:
https://algo.cs.uni-frankfurt.de/lehre/oau/sommer23/oau23.shtml

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