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Exercise 7

Issued: 11.12.2018
Due: 18.12.2018

Exercise 7.1. *Variant of Valiant's Trick* (3 + 3 = 6 Points)

We study permutation routing on the d-dimensional hypercube with $n = 2^d$ nodes. Recall that Valiant's trick chooses routing paths of dilation $D \leq 2d$ and congestion $O(d/\log d)$, w.h.p., for any given permutation.
In this exercise, we show that the dilation can be reduced to $D \leq d$ while keeping the congestion bound $C = O(d/\log d)$, w.h.p.

Consider the following variant of Valiant's trick: For each packet p with source s_p and target t_p, two intermediate destinations $v_p^{(1)}$ and $v_p^{(2)}$ are chosen as follows: Node $v_p^{(1)}$ is picked independently and uniformly at random from V. The node $v_p^{(2)}$ is defined to be the bit-wise complement of $v_p^{(1)}$. This procedure gives two alternative paths for packet p: The first path leads from s_p via a bit-fixing path to $v_p^{(1)}$ and then via a bit-fixing path to t_p. The second path is defined analogously using the intermediate destination $v_p^{(2)}$ instead of $v_p^{(1)}$. Each packet is then sent along the shorter of its two alternative paths.

a) Show that the dilation D is at most d.

b) Show that the congestion C is upper-bounded by $O(d/\log d)$, w.h.p.

Exercise 7.2. *Increasing the Lower Bound* (5 Points)

We study permutation routing on the d-dimensional hypercube with $n = 2^d$ nodes. Suppose packets should be sent along bit-fixing paths. Show that there exists a permutation such that the number of paths that share an edge is $\Omega(\sqrt{n})$.

Hint: Consider the bit reversal permutation defined by

$$rev(x_{d-1}, x_{d-2}, \ldots, x_1, x_0) = (x_0, x_1, \ldots, x_{d-2}, x_{d-1}).$$

How many nodes (as a function of n) have a bit label ending with $\lfloor d/2 \rfloor$ zeroes? How does the bit-fixing path of a packet starting from such a node look like?

Please turn over!

Exercise 7.3. *Push* (2 + 3 = 5 Points)

We study the Push protocol for rumor spreading:
In every round independently, each node that has the message sends (pushes) it to a neighbor picked uniformly at random.

a) Show that the Push protocol takes $\Omega(\log n)$ many rounds to inform all nodes for all graphs.

b) Show that in expectation, the Push protocol takes $\Omega(n \log n)$ many rounds to inform all nodes on a star graph.

   Hint: Coupon-Collection

Exercise 7.4. *Push-Pull* (2 + 4 = 6 Points)

We study the Push-Pull protocol for rumor spreading:
In every round independently, each node that has the message sends (pushes) it to a neighbor picked uniformly at random. Additionally, each node that does not have the message asks a neighbor (again, picked independently and uniformly at random) for news.

a) Show that the Push-Pull protocol takes $O(1)$ many rounds to inform all nodes on a star graph.

b) Give an example of a graph with constant diameter where the Push-Pull protocol takes $\Omega(n)$ rounds to inform all nodes in expectation.

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