Theory of Distributed Systems

Winter Term 2021/22

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Exercise 7

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Please submit your solution in PDF format by sending an email to {schmalhofer,varricchio}@em.uni-frankfurt.de. Make sure that your solution reaches us before 8:15 am! Solutions are discussed on Dec 17th, 10:00h - 12:00h (Zoom Meeting-ID: 963 6309 6725, same password as lecture material).

Exercise 7.1. *Dimension-By-Dimension on Meshes* (7 = 2 + 2 + 3 Points)

a) Prove that for every permutation routing problem on the mesh $M(\ell, 1)$, dimension-by-dimension routing terminates in $O(Diam(G))$ rounds.

b) Consider dimension-by-dimension routing on the mesh $M(\ell, 2)$. Show that if routing targets are no longer a permutation (i.e., a general function $f : V \to V$, where $f(v)$ is the target of node v), then time complexity can be $\Omega(Diam(G)^2)$.

c) Show that for the mesh $M(\ell, 3)$, there are permutations such that dimension-by-dimension routing takes $\Omega(Diam(G)^2)$ rounds.

Exercise 7.2. *Dimension-By-Dimension on Hypercubes* (6 = 3 + 3 Points)

Consider dimension-by-dimension permutation routing in a hypercube $M(2, d)$ with $n = 2^d$ nodes. Assume nodes are labeled by bit strings of length d. Moreover, the labeling is such that any pair of nodes is connected by an edge if and only if their bit strings differ in exactly one position: If $k \in \{1, \dots, d\}$ is the position in which two nodes differ, the corresponding edge is called *k-edge*.

a) Show that, for every permutation, any k-edge is used in one direction by at most $\min\{2^{d-k}, 2^{k-1}\}$ many packets.

b) Show that there are permutations such that delivering all packets takes $\Omega(\sqrt{n})$ rounds.

Exercise 7.3. *Indirect Networks* (5 Points)

Let $G = (V, E)$ be a graph with two special subsets $I \subseteq V$ and $O \subseteq V$ called the inputs and the outputs, respectively. Suppose $|I| = |O|$. Such a network is called *indirect network*. A path system $\mathcal{W}$ for an indirect network contains a path $P_{u,v}$ from every input $u \in I$ to every output $v \in O$. A permutation routing problem on indirect networks is given by a bijective function $\pi : I \to O$ (rather than a complete permutation $\pi : V \to V$).

Generalize the lower bound of Theorem 23 towards permutation routing problems on indirect networks. In particular, prove a lower bound in terms of n, Δ, and r, where $n = |I|$, Δ is the maximum degree of G and r denotes the ratio between the number of nodes and the number of inputs, i.e., $r = |V|/n$. For $r = 1$, your bound should be identical to the one in Theorem 23.

Exercise 7.4. *Hamiltonian Cycles* (4 Points)

Prove or disprove:
The hypercube $M(2, d)$ has a Hamiltonian cycle for every $d \geq 2$.

Reminder: A Hamiltonian cycle in a graph $G = (V, E)$ is a cycle in G that contains every vertex in V exactly once.

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